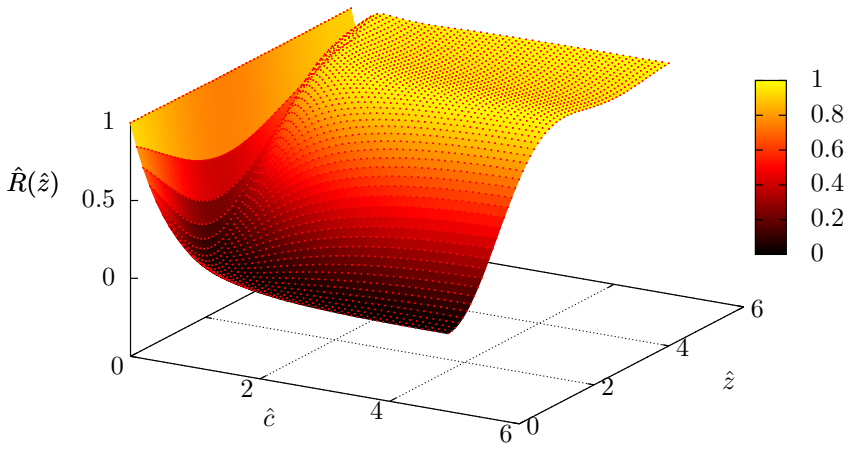
# Dissertation

zur Erlangung des akademischen Grades
Doktor der Naturwissenschaften
(Dr. rer. nat.)

UNIVERSITÄT DUISBURG ESSEN

# Highly imbalanced Fermion-Fermion mixtures in one dimension

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Hiermit versichere ich an Eides statt, die vorliegende Dissertation selbstständig, ohne fremde Hilfe und ohne Benutzung anderer als den angegebenen Quellen angefertigt zu haben. Alle aus fremden Werken direkt oder indirekt übernommenen Gedanken sind als solche gekennzeichnet. Die vorliegende Dissertation wurde in keinem anderen Promotionsverfahren eingereicht. Mit dieser Arbeit strebe ich die Erlangung des akademische Grades "Doktor der Naturwissenschaften" (Dr. rer. nat.) an.

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**Teile der vorliegende Arbeit sind bereits in den unter 1. und 2. genannten Manuskripten veröffentlicht. Die unter 3. aufgeführte Veröffentlichung ist nicht Bestandteil dieser Arbeit.**

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# Zusammenfassung

Wir untersuchen eindimensionale Systeme von Spin-$\frac{1}{2}$ Fermionen im Rahmen quantenmechanisch exakt lösbarer Modelle.

Im ersten Teil wird das eindimensionale Modell von Spin-$\frac{1}{2}$ Teilchen mit abstoßender Kontaktwechselwirkung studiert. Für die Eigenfunktionen des entsprechenden Hamilton Operators, welche mittels des Bethe-Ansatzes exakt konstruiert werden können, wird eine Neuformulierung präsentiert. Insbesondere für den Fall, dass von einer Teilchensorte (Minoritäts-Fermionen) nur sehr wenige Teilchen vorhanden sind, während die Anzahl der Teilchen der anderen Sorte (Majoritäts-Fermionen) im thermodynamischen Limes gegen unendlich geht, weist die Neuformulierung der Vielteilchenwellenfunktion eine handliche Form auf. Für den Grundzustand ist die nichtwechselwirkende Impulsverteilung der Majoritäts-Fermionen durch einen Fermi-See gegeben. Ausgehend hiervon werden nun Anregungen studiert, bei denen die Minoritäts-Fermionen einen beliebigen Zustand innerhalb des Fermi-Sees einnehmen können.

Für den Fall, dass nur ein Minoritäts-Fermion anwesend ist, lassen sich die exakten Eigenfunktionen des Systems als Determinante schreiben. Dies erlaubt es exakte Ausdrücke für Erwartungswerte sowie die Dichte-Dichte Korrelationsfunktion im thermodynamischen Limes zu berechnen. Des Weiteren können geschlossene Ausdrücke für die Einteilchen Greensfunktion des Minoritäts-Fermions hergeleitet werden. All diese Größen zeigen eine sensible Abhängigkeit von dem Impuls des Minoritäts-Fermions. Insbesondere wird gezeigt, dass die Greensfunktion im Tonks-Girardeau Regime unendlich starker Wechselwirkung, einen Übergang von der Greensfunktion von hardcore Bosonen, zu jener von freien Fermionen zeigt. Dieser Übergang manifestiert sich in einem algebraischen Abfall der Greensfunktion für große Abstände.

Sind nun zwei Minoritäts-Fermionen anwesend, so nehmen die Eigenfunktionen eine komplizierte Gestalt an. Dennoch können auch für diesen Fall Dichte-Dichte Korrelationsfunktionen exakt berechnet werden. Die Analyse der Gesamtenergie des Systems erlaubt es Beiträge zu identifizieren, welche eine natürliche Interpretation als effektive Wechselwirkungsenergie der beiden Minoritäts-Fermionen aufweisen.

Im zweiten Teil der Arbeit werden eindimensionale Systeme bestehend aus zwei Teilchensorten mit unterschiedlichen Massen untersucht. Es wird gezeigt, dass für eine Reihe von Wechselwirkungspotenzialen sowie für bestimmte Beziehungen zwischen den Massen und Kopplungskonstanten, Operatoren zur Teilchen Erzeugung/Vernichtung für solch ein System exakt konstruiert werden können.

# Abstract

In the framework of exactly solvable quantum many-body systems we study models of interacting spin one-half Fermions in one dimension.

The first part deals with systems of spin one-half Fermions which interact via repulsive contact interaction. A reformulation of the Bethe-Ansatz solvable many-body wave function is presented. This simplifies considerably the calculations for the highly imbalanced case, where very few particles of one species (minority Fermions) are present. For the other particle species (majority Fermions) the thermodynamic limit is taken. We assume the majority Fermions to be in the ground state such that their non-interacting momentum distribution is a Fermi-sea. Upon this we consider excitations where the particles of the minority species may occupy an arbitrary state within the Fermi-sea.

In the case of only a single minority Fermion, the many-body wave function can be expressed as a determinant. This allows us to derive exact thermodynamic expressions for several expectation values as well as for the density-density correlation function. Moreover it is possible to find closed expressions for the single particle Green's function. All of the above mentioned quantities show a non-trivial dependence on the minority particle's momentum. In particular the Green's function in the Tonks-Girardeau regime of hardcore interaction is shown to undergo a transition from the one of impenetrable Bosons to that of free Fermions as the extra particle's momentum varies from the core to the edge of the Fermi-sea. This transition becomes manifest in an algebraic asymptotic decay of the Green's function.

If two minority Fermions are present, the many-body wave function turns out to be more complicated. Nevertheless it is possible to derive exact expressions for the two and the three particle density-density correlation functions. Furthermore we calculate the system's total energy and based on that, identify terms which have a natural interpretation as effective interaction energy for the two minority Fermions in the presence of the Fermi-sea.

The second part is devoted to the study of one-dimensional systems consisting of two fermionic particle species with different masses. We show that for specific kinds of interaction potentials and for certain relations between the masses and the coupling constants, the particle creation and annihilation operators of such a system can be constructed exactly.

# Contents

# Chapter 1

# Introduction

The study of one-dimensional quantum many-body systems has a long history dating back to the founding days of quantum mechanics [1, 2]. In those times however, the interest in systems with one-dimensional geometries was mostly academic and their investigation a niche of theoretical physics. This gradually changed over the decades. Whereas during the sixties and seventies their study still was predominantly of theoretical nature [3, 4, 5, 6, 7, 8, 9], the first realizations of quasi one-dimensional electron gases in the laboratory [10, 11, 12, 13] during the eighties began to attract the interest of a broader community. Since then a steadily increasing amount of attention has been devoted to the subject. Nowadays there are extensive ongoing investigations from both, the experimental [14, 15, 16, 17] as well as the theoretical [18, 19] angle, making one-dimensional quantum many-body systems a topic of modern research in its own right. We briefly comment on the most important experimental and theoretical aspects of one-dimensional systems.

Over the past few decades significant progress in the preparation of one-dimensional systems in the laboratory has taken place. By now they can be realized in experimental setups such as carbon nanotubes [20, 21], semiconductor heterostructures [14] or ultracold quantum gases [22, 15, 17]. In particular the framework of ultracold atoms provides a unique way of creating and engineering one-dimensional systems. In that manner the creation of identical Bosons [22, 15, 16, 17] and even systems with different particle species [23] in one-dimensional geometries became feasible. Moreover it allowed long-standing exactly solvable models, such as the Tonks-Girardeau gas [24, 15, 17] of impenetrable Bosons, to be synthesized for the first time. This establishes the link between the more academic subject of exactly solvable models and recent experimental realizations of one-dimensional systems in the laboratory and leads to a new testing ground for models of quantum many-body physics.

From a theoretical point of view one-dimensional systems are exceptional for two reasons. On the one hand, their theoretical description is challenging. Rather than being described by the conventional and more familiar Fermi-liquid theory, electrons in one dimension are a Luttinger-liquid [25, 26]. The breakdown of the Fermi-liquid theory comes along with peculiar features of one-dimensional electron liquids such as

spin-charge separation [27]. On the other hand, it is known that for certain kinds of interaction potentials the eigenfunctions of one-dimensional interacting many-body systems can be found exactly [28, 29, 30, 31, 32, 33]. This partially compensates the above mentioned problems and has attracted in the past attention not only from the physical but also from the mathematical side.

An intuitive argument that might explains the difference between one and higher dimensional systems is the following: Consider the scattering of two particles with opposite momentum that interact repulsively. Schematically this scenario is depicted in Fig. 1.1. In two or three dimensions a direct collision will be avoided by changing the direction of the trajectory of each particle. A small interaction will thus slightly change the original trajectory when the particles come close to each other. If the same situation takes place in one dimension, the lack of spatial freedom forces the particles to collide. Loosely speaking, the particles cannot avoid each other. Therefore, even for small interactions the motion of one particle will strongly affect the motion of the other and vice versa. Now, if more than just two particles are present, the dimensional reduction crucially affects the nature of the elementary excitations. In order for a particle to be able to move it will have to push its neighbors. The latter in turn need to push their neighbors and so on. The motion of the particles is thus strongly correlated. Intuitively this might be identified as the reason why the elementary excitations in one-dimensional electron systems are collective rather than single particle like.

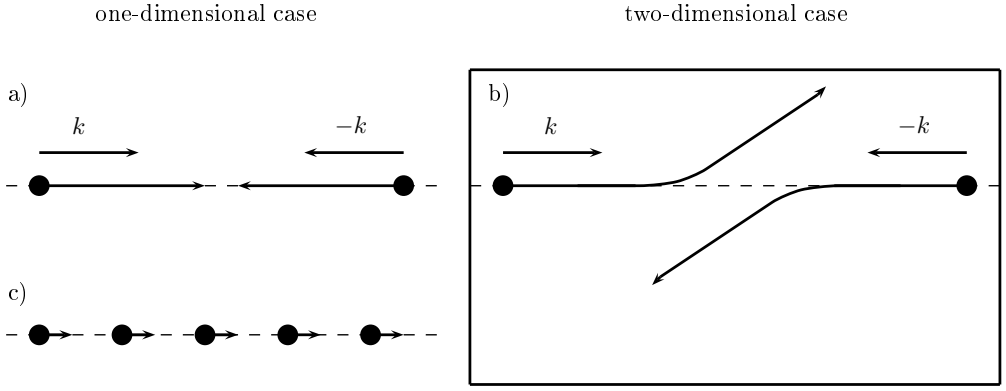


Figure 1.1: Scattering of two particles with opposite momentum $k$ in one and two dimensions. Whereas in one dimension (Fig. a)) the two particles unavoidably collide, a direct collision is prevented in the two dimensional case (Fig. b)). In one dimension each particle has to push its neighbor in order to move (Fig. c)). This leads to elementary excitations that are collective.

## 1.1 Experimental realization of one-dimensional systems

Nowadays there are several experimental setups which allow the realization of one-dimensional systems in the laboratory. The most popular among these are semiconductor

heterostructures, carbon nanotubes and ultracold quantum gases.

Systems that exhibit a one-dimensional geometry are often summarized under the term *quantum wires* [34]. By this one usually refers to a conducting system for which the spatial degrees of freedom in two (transverse) directions are drastically reduced such that a particle effectively can move only along one (longitudinal) direction. The strong transverse confinement leads to widely separated energy levels and hence to a discrete energy spectrum for the motion along these directions. If the lowest transverse excitation exceeds all other energies in the system, only the ground state regarding these directions will be occupied and the dynamics of the system is in fact one-dimensional. In the sequel we discuss two experiments that demonstrate the peculiar dynamical features of one-dimensional systems.

### 1.1.1 One-dimensional systems in cold atom gases

The ground breaking experiments undertaken by Cornell, Wieman and Ketterle and their co-workers on the trapping and cooling of atoms led to the experimental realization of ultracold quantum gases and culminated in the first realization of a Bose-Einstein condensate in the laboratory [35, 36] in 1995. Nowadays these techniques can be used to realize and test various models of condensed matter physics [37]. In particular it is possible to create ultracold atom gases in one-dimensional geometries [22, 15, 38, 16, 17, 39, 40].

Starting from a Fermi or Bose gas in a three-dimensional trap, one-dimensional systems can be realized by superimposing a two-dimensional optical lattice in the $xy$-plane. Schematically the experimental setup is shown Fig. 1.2. The optical lattice is formed by two opposing laser beams adjusted such that they form a standing wave. For sufficiently strong lattice potential the particles are confined in elongated cigar-shaped geometries along the $z$ direction leading to effectively one-dimensional systems. The weakly varying potential in the $z$ direction arises due to the superimposed three-dimensional trapping potential.

A milestone for the experiments of one-dimensional ultracold quantum gases was the creation of identical Bosons in the Tonks-Girardeau regime by Kinoshita and co-workers [15]. In a further experiment [17] Kinoshita et al. prepared a system of two out-of-equilibrium clouds of Bosons which collide inside a one-dimensional trap. The left picture in Fig. 1.3 shows a cartoon which schematically depicts this scenario. To study the dynamics of the system, the experiment is repeated various times, always starting from the same initial condition and letting the system evolve. Then at a certain moment, the atoms are released from the trap and are allowed to expand freely before an absorption image is taken. On the right hand side of Fig. 1.3 the resulting images for several times are shown. It shows the oscillation of the two atom clouds during one cycle. The measured absorption images allow the momentum distribution of the atoms to be extracted. Quite remarkably the analysis of the experimental data showed that, even after thousands of oscillations, no redistribution of the momenta, which would indicate a thermalization of particles, takes places. The authors conjectured that this unusually slow decay of the system might arises due to the close relation of the experimentally

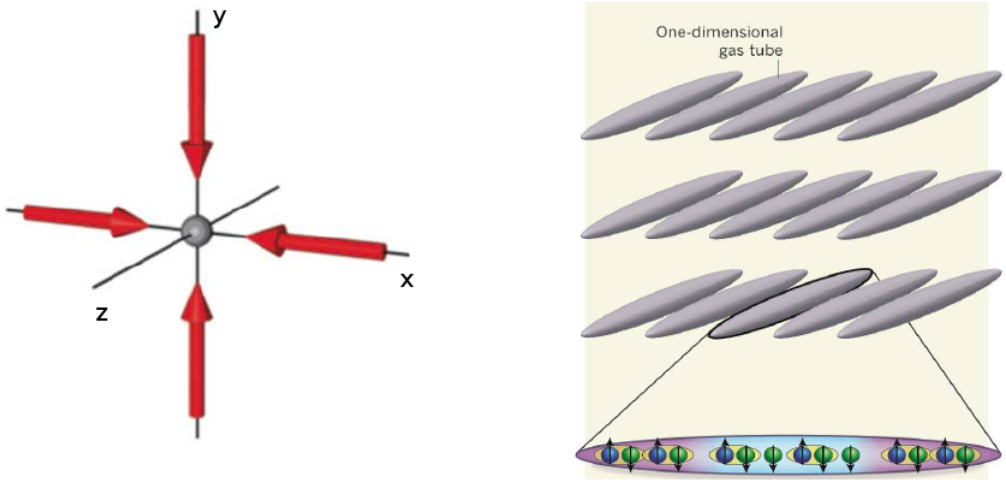


Figure 1.2: Schematic setup for creating one-dimensional systems of ultracold atoms. Left): The superposition of two opposing laser beams (indicated by the red arrows) leads to a lattice potential for the trapped atoms. Adapted from Ref. [37]. Right: For sufficiently strong lattice potential the particles are confined in one-dimensional elongated tubes. Taken from Ref. [19].

realized system with the exactly solvable Lieb-Liniger model [3] of identical Bosons with contact interaction.

### 1.1.2 Tunneling between parallel quantum wires

In Ref. [14] Barak et al. studied the momentum and energy resolved tunneling between parallel quantum wires realized in a semiconductor heterostructure. Their results show the differences for the relaxation process between particles and holes. Whereas an excited particle is allowed to relax through multiple particle-hole excitations, the relaxation of a hole is suppressed due to the conservation of energy and momentum. Our presentation of the experiment follows the one given in Ref. [18].

The schematic setup is depicted in Fig. 1.4 (a). In the experiment electrons or holes are injected from one quantum wire (lead 1) into a parallel grounded quantum wire and are extracted at lead 2. A magnetic field applied perpendicular to the plane formed by the quantum wires and a bias applied to the two leads allow the dispersion relations of lead 1 and lead 2 to be shifted relative to the dispersion relation of the lower wire. The momenta of the injected and extracted particles is controlled by the magnetic field. Within the experimental setup holes with momenta below the Fermi-momentum $k_{\mathrm{F}}$ were injected from lead 1 into the lower wire for magnetic fields below $B \approx 4.77$ Tesla (the injection of electrons with $k < k_{\mathrm{F}}$ is suppressed, since the corresponding states are occupied in the lower wire). The holes in the lower wire are then extracted at the second lead. The right picture of Fig. 1.4 shows the measured currents at the two leads. It

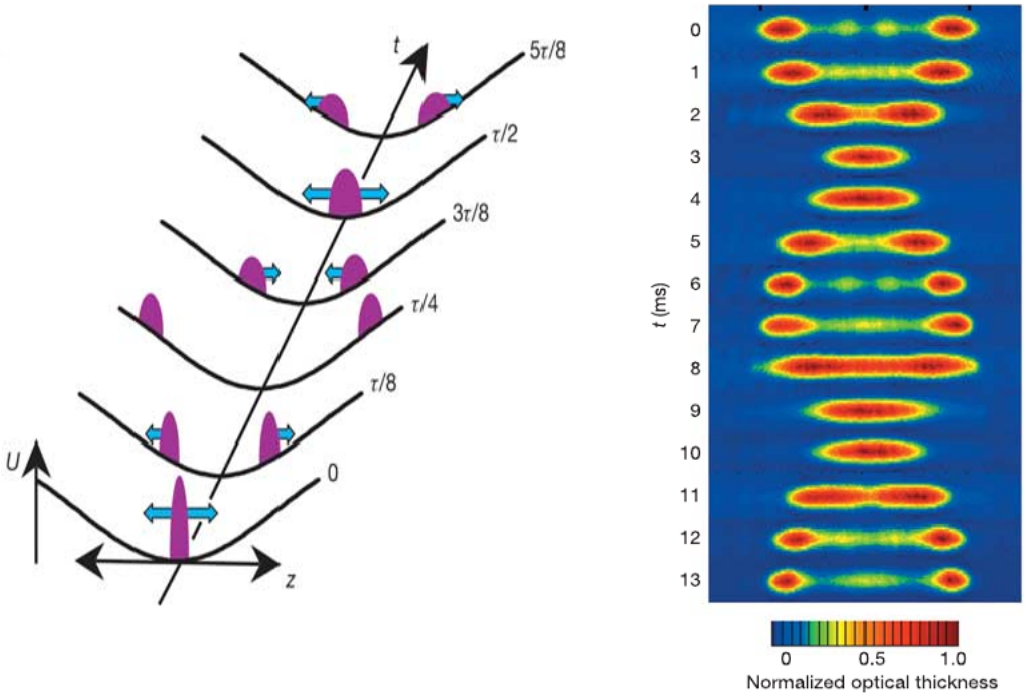


Figure 1.3: Left: Cartoon of two colliding atom clouds with opposite momentum confined in a one-dimensional trapping potential $U(z)$. Right: Absorption images of two oscillating atom clouds obtained in the experiment. Taken from Ref. [17].

is seen that if holes are injected, the currents at lead 1 and at lead 2 coincide. This indicates that the injected holes can not relax, see Fig. 1.4 (b). If on the other hand the magnetic field is higher than 4.77 Tesla, electrons with momenta $k > k_{\mathrm{F}}$ are injected from lead 1 into the lower wire. In this case the resulting current at lead 2 exceeds the injected one at lead 1. This is explained by the relaxation of the injected electrons through the generation of multiple particle-hole pairs. The creation of particle-hole pairs leads to additional electrons above the Fermi-edge, see Fig. 1.4 (c). Whereas the latter ones are allowed to tunnel from the lower wire into the second lead, the holes remain in the lower wire. Since only the electrons are extracted, the current in the second lead exceeds the injected one at lead 1.

## 1.2 Exactly solvable models

The history of exactly solvable models in quantum mechanics starts with the pioneering work of H. Bethe who in 1931, just after the formulation of quantum mechanics, found the exact eigenfunctions of the one-dimensional isotropic Heisenberg-model [1, 2]. The Ansatz he used is nowadays known as *Bethe-Ansatz* and has proven to be appli-

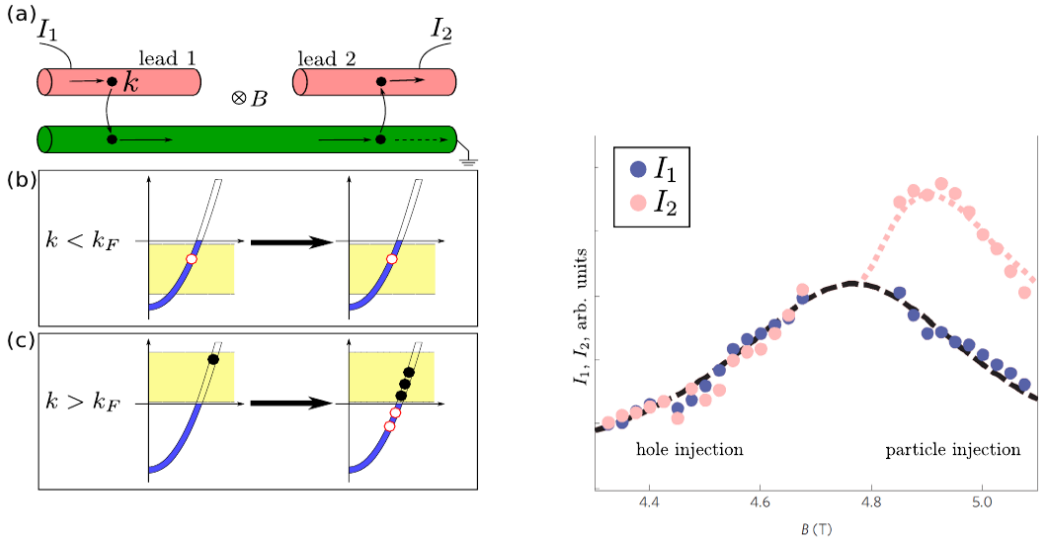


Figure 1.4: Left: Schematic setup of the experiment (Fig. (a)). Particles or holes are injected from lead 1 into the lower wire and are extracted at lead 2. The magnetic field controls the momenta of the injected and extracted particles. Whereas holes can not relax (Fig.(b)), particles can relax via the excitation of multiple particle-hole pairs (Fig. (c)). Right: Currents $I_1$ and $I_2$ measured at the leads as function of the magnetic field. For hole injection the currents coincide. Injection of electrons leads to a current $I_2$ that exceeds the current in $I_1$. Taken from Ref. [18].

cable to either two-dimensional lattice models of classical mechanics [41, 42, 43], or one-dimensional quantum mechanical systems [1, 3, 6, 7]. Beside the Bethe-Ansatz there are nowadays several other approaches to find and treat exactly solvable models such as the asymptotic Bethe-Ansatz [8] or the quantum inverse scattering method [28]. This leads to a rich variety of fields for which exact solutions of many-body systems are available: From lattice models of classical statistical mechanics [43], over discrete and continuous quantum mechanical systems [1, 3, 8], to relativistic models of quantum field theory [44, 45]. The topic with its several issues has been summarized in a series of monographs [46, 28, 29, 30, 47, 48]. In what follows we give a brief account of continuous quantum mechanical systems which are of particular interest for our purposes.

We consider a non-relativistic one-dimensional quantum many-body system consisting of $N$ particles with mass $m$ which interact via a two-body potential $V(x)$. In coordinate representation the corresponding Hamiltonian reads

$$H_N(\mathbf{x}) = -\frac{\hbar^2}{2m}\sum_{n=1}^{N}\frac{\partial^2}{\partial x_n^2} + \sum_{n\neq j}^{N} V(x_n - x_j)\ . \tag{1.1}$$

Here $x_1, \ldots, x_N$ denote the coordinates of the $N$ particles on the real axis. Customarily the units are chosen such that $\hbar = 1$ and $m = 1/2$. We will follow that practice, since

it helps to keep formulas as simple as possible. Before considering explicit examples, we clarify what we mean by the term "exactly solvable": A system is called exactly solvable, if a complete set of eigenfunctions of the underlying Hamiltonian (1.1) can be constructed exactly. Knowing the exact eigenfunctions of a model, their completeness is usually assumed. However, proofs are only known for some special cases [49].

Sometimes exactly solvable models are also referred to as "integrable". However, integrability is a concept born of classical mechanics. There a system with a $2N$-dimensional phase space is called integrable, if it possesses $N$ independent constants of motion. This means that the Poisson brackets $\{\bar{H}, O_j\} = 0$ of the corresponding observables $O_j$, $j = 1, \ldots, N$ with the classical Hamiltonian $\bar{H}$ vanishes and that furthermore the observables are in involution i.e. $\{O_j, O_l\} = 0$. In particular integrability is a constructive method: Knowing the constants of motion, the solution to the Hamiltonian equations can be given. In analogy one might call the quantum many-body system described by Eq. (1.1) integrable, if there exist $N$ mutually commuting operators which also commute with the Hamiltonian $H_N$. However, in marked contrast to the classical case, it is not clear how to construct the exact eigenfunctions of the Hamiltonian based on this property. Since we deal with quantum mechanical systems, we shall avoid the usage of the terminology integrable.

Two archetypal interaction potentials for which the eigenfunctions of the Hamiltonian can be constructed exactly are the short range $\delta$-potential $V(x) \propto \delta(x)$ and the long range inverse square potential $V(x) \propto 1/x^2$. Depending on the interaction potential under consideration there are several methods available to treat the system.

### 1.2.1 Bethe-Ansatz for particles with contact interaction

For particles with contact interaction we write the potential in the Hamiltonian (1.1) as $V(x) = 2c\delta(x)$. The eigenfunctions of $H_N$ can be constructed by means of the Bethe-Ansatz [3, 50, 4, 5, 6, 7]. In this context, Bethe-Ansatz is also referred to as the *coordinate Bethe-Ansatz*. Its essential idea is the following: Since the interaction potential is local, the eigenfunctions are linear superpositions of plane waves. The coefficients of the linear combination can be determined by applying the matching boundary conditions for the wave function. Depending on the statistics of the particles, the procedure of finding the coefficients is a more or less difficult task.

The simplest case is the one of identical particles. Since identical Fermions are, due to Pauli's-principle, forbidden to be at the same position, a $\delta$-potential remains unseen by them. Therefore only the bosonic case is relevant. The corresponding model of identical Bosons interacting via a repulsive $\delta$-potential has been solved by Lieb and Liniger [3] and is therefore also referred to as Lieb-Liniger model. They found the exact eigenfunctions and calculated the ground state properties of the system. Starting from the eigenfunctions, the usual procedure for a further treatment is as follows:

The first step is to subject the wave function to periodic boundary conditions. This results in quantization rules for the quasi-momenta in form of transcendental equations

which are known as *Bethe-Ansatz equations* [29]. For identical Bosons they read [3]

$$k_n L = 2\pi m_n + \sum_{j\neq n}^{N} \theta(k_n - k_j) \quad , \quad n = 1, \ldots, N \ , \tag{1.2}$$

where the integers $m_n$, $n = 1, \ldots, N$ are the quantum numbers, $L$ is the size of the system and $\theta(k) = -2\arctan(k/c)$ is the phase shift for the $\delta$-potential. This is a set of $N$ coupled algebraic equations for the quasi-momenta $k_n$. Its solution determines the energy spectrum of the system.

Next, the step to the infinite system is taken. In the thermodynamic limit, where $N, L \to \infty$ such that the particle density $N/L$ remains finite, the quasi-momenta for the ground state are distributed with a symmetric density $\rho(k)$ in between $\pm q$. The Bethe-Ansatz equations translate to a Fredholm type of integral equation for $\rho(k)$

$$\rho(k) = \frac{1}{2\pi} + \frac{c}{\pi} \int_{-q}^{+q} dk' \frac{\rho(k')}{(k-k')^2 + c^2} \ , \tag{1.3}$$

where the bounds of the integral are determined by the normalization condition of $\rho(k)$ to the particle number $N$. Based on its solution the zero temperature thermodynamics of the system can be calculated [3, 29]. However, an analytical solution of Eq. (1.3) is available only in the limit of vanishing or infinitely strong interaction strength.

A finite temperature treatment is also possible. The pioneering approach in this context was established by Yang and Yang [51] and the corresponding techniques are summarized under the item *thermodynamic Bethe-Ansatz*. A comprehensive review can be found in Takahashi's book [29]. Further important quantities such as Green's functions or density-density correlation functions have also been studied. In particular the Tonks-Girardeau regime of infinitely strong interaction strength attracted the attention [52, 53, 54, 55, 56, 57]. In this limit the interaction potential becomes impenetrable and the eigenfunctions relate to those of free Fermions via the *Bose-Fermi mapping*. The wave functions of the two systems relate through [24, 58]

$$\Psi_{\text{impentrable Bosons}}(x_1, \ldots, x_N) = \prod_{j<l}^{N} \mathrm{sgn}(x_j - x_l) \Psi_{\mathrm{F}}(x_1, \ldots, x_N) \ , \tag{1.4}$$

where $\mathrm{sgn}(x) = x/|x|$ denotes the sign-function and $\Psi_{\mathrm{F}}(x_1, \ldots, x_N)$ is the wave function of free Fermions i.e. a Slater determinant.

The generalization of the Bethe-Ansatz from identical particles to particles with internal degrees of freedom such as spin one-half Fermions turns out to be more complicated. The reason for this is that the many-body wave function transforms according to a higher dimensional representation of the symmetric group. For systems with different particle species Bethe-Ansatz is also referred to as *nested Bethe-Ansatz*. The problem of spin one-half Fermions has been solved gradually. First McGuire constructed the exact

ground state eigenfunctions for the case where a single spin-up particle interacts repulsively [4] or attractively [5] with an arbitrary number of spin-down particles. This has been extended by Flicker and Lieb [50] to the case of two spin-up particles with repulsive interaction. The general case for an arbitrary number of spin-up and spin-down particles was solved by Yang [6] and Gaudin [7] which established the nested Bethe-Ansatz. A more detailed discussion of the eigenfunctions for spin one-half Fermions with contact interaction will be given in the next chapter. Later the nested Bethe-Ansatz was applied to systems of Fermions and/or Bosons with higher internal degrees of freedom [59, 8, 60].

Although more complicated, the further treatment of all these systems follows along the same lines as the above sketched approach for identical Bosons. The more complicated form of the eigenfunctions lead to involved expressions for the Bethe-Ansatz equations and the resulting integral equations. In particular the problem of calculating Green's functions and correlation functions is solved only in some special cases [28, 52, 61].

### 1.2.2 Long-range interaction potentials

For continuous models the applicability of the Bethe-Ansatz in its original form is restricted to systems with local interaction. Nevertheless there are also long-range interaction potentials for which the eigenfunctions of the Hamiltonian (1.1) can be constructed exactly. Paradigms of those are the inverse square potential $V(x) = g/x^2$ or its periodic version $V(x) = gb^2/\sin^2(xb)$, the trigonometric Calogero-Moser-Sutherland (CMS) model.

The problems connected with long-range interaction potentials have partly been overcome via the *asymptotic Bethe-Ansatz*, introduced by Sutherland [8, 9]. Its main assumption is that the form of the Bethe-Ansatz equations as in Eq. (1.2) still holds for non-local interactions provided that the multi-particle scattering process is reducible to a sequence of two particle scatterings only. Assuming its validity, Sutherland calculated by means of the asymptotic Bethe-Ansatz the ground state thermodynamics of the inverse square potential even without knowledge of the exact eigenfunctions [9]. For the inverse square potential the correctness of the asymptotic Bethe-Ansatz has later been proven [62, 63].

In the case of the trigonometric CMS-model, the exact eigenfunctions can be constructed via Jack-polynomials [64]. Although the latter ones are known only recursively, a detailed knowledge of their properties allowed the evaluation of closed expressions for the time-dependent density-density correlation function [65]. Also the time-dependent single particle Green's function could be calculated for specific values of the interaction strength [66].

The basic assumption of the asymptotic Bethe-Ansatz about the factorization of the multi-particle scattering process into a sequence of two particle scatterings was identified as the fundamental property of exactly solvable models. Mathematically, it is expressed in terms of the Yang-Baxter equation. With the Yang-Baxter equation as a starting point a systematic approach for the quest and solution of one-dimensional models with long and short range interactions was established, which is known as *Quantum inverse*

*scattering method* [28]. Since the approach yields an algebraic method of constructing the exact eigenfunctions, it is also referred to as *algebraic Bethe-Ansatz*. Rather recently a new approach based on the explicit construction of particle creation and annihilation operators for one-dimensional systems has been developed [67]. Its application to one-dimensional systems of spin one-half Fermions is one of the topics of this work.

## 1.3 Outline

This thesis is divided into two parts. During the first part, we study spin one-half Fermions in one dimension with repulsive contact interaction. In the second part, we discuss a method for constructing particle creation and annihilation operators for one-dimensional systems of spin one-half Fermions.

In Chapter 2, we introduce the basic model for spin one-half Fermions with $\delta$-interaction. The exact construction of the eigenfunctions by means of the Bethe-Ansatz is discussed in Sec. 2.1. Afterwards we present in Sec. 2.2 a reformulated form of the exact eigenfunctions and discuss their symmetries. The corresponding Bethe-Ansatz equations as well as the thermodynamic limit are the topic of Sec. 2.3. In Sec. 2.4, the one-dimensional Hubbard model is considered.

Chapter 3 is devoted to the study of McGuire's model. This corresponds to the highly imbalanced case of the model introduced in Chapter 2. A more detailed discussion of the Bethe-Ansatz equations for this case as well as the necessary ingredients are provided in Sec. 3.1. In Sec. 3.2, we calculate several expectation values as the energy-shift, the interaction energy and the kinetic energy of the extra particle. The equal time single particle Green's function is discussed in Sec. 3.3. Particular emphasis is devoted to the hardcore limit, where we relate the Green's function to solutions of Painlevé transcendental equations. Subsequent we study in Section 3.4 the density-density correlation function. The survival probability and the local density of states are discussed in Sec. 3.5.

In Chapter 4, we consider the case where two spin-up particles interact with a Fermi-sea. The necessary ingredients are collected in Sec. 4.1. We study the energy-shift in Sec. 4.2. In Sec. 4.3 the two and the three particle density-density correlation functions are considered. A summary of Part I is given in Sec. 4.4.

The second part deals with the exact construction of particle creation and annihilation operators for systems of spin one-half Fermions in one dimension. In Sec. 5.1, we introduce the approach by reviewing the method as it was developed for identical Fermions. Section 5.2 provides the general framework. In Sec. 5.3 we discuss the exact construction of particle creation and annihilation operators in coordinate representation for the case of spin one-half Fermions with different masses and coupling constants. A few applications of the method are discussed in Sec. 5.4. We conclude in Sec. 5.5. An overall conclusion of the thesis is given in Chapter 6.

To keep the continuity of the presentation, several longer calculations are presented in the Appendices A and B.

# Part I

# Fermions with contact interaction

# Chapter 2

# Basic concepts

The purpose of this chapter is twofold: Its first aim is to give an elementary introduction into the model considered during the first part. Its second goal is to provide the fundamental concepts and results needed in Chapters 3 and 4. The outline of the chapter is as follows:

In Sec. 2.1, we introduce the model and briefly recapitulate its exact solution by means of the Bethe-Ansatz. In Sec. 2.2, we present a reformulation of the exact eigenfunctions which serves as starting point for the results presented in Chapters 3 and 4. The Bethe-Ansatz equations and the thermodynamic limit are discussed in Section 2.3. In Sec. 2.4, we consider the one-dimensional Hubbard model.

## 2.1 The Model and its exact solution

We consider $N + M$ non-relativistic Fermions in one dimension which interact via a repulsive $\delta$-potential. The system is made up of two sorts of Fermions. To the first Fermion species we refer to as spin-down Fermions and we assume $N$ particles of this type to be present. The remaining $M$ particles belong to the second species and we refer to them as spin-up Fermions.

The system is governed by the stationary Schrödinger equation

$$H\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda}) = E\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda}) \ , \tag{2.1}$$

where $E$ denotes the energy corresponding to the eigenfunction $\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})$ and the Hamiltonian reads

$$H = -\left(\sum_{n=1}^{N}\frac{\partial^2}{\partial x_n^2} + \sum_{m=1}^{M}\frac{\partial^2}{\partial y_m^2}\right) + 4c\sum_{n=1}^{N}\sum_{m=1}^{M}\delta(x_n - y_m) \ . \tag{2.2}$$

Here the units are chosen such that $\hbar = 1$ and furthermore all masses are equal to $1/2$. This is useful in order to unburden the notation and keeping formulas as simple and transparent as possible. Unless otherwise stated, we will use this convention throughout the rest of the thesis.

We consider the case of repulsive interaction and hence $c \geq 0$. The factor 4 is introduced for convenience. The coordinates $\mathbf{x} = \{x_n\}_{n=1,\ldots,N}$ and $\mathbf{y} = \{y_m\}_{m=1,\ldots,M}$ in Eqs. (2.1) and (2.2) refer to the positions of the spin-down and spin-up Fermions, respectively. As indicated in Eq. (2.1), the eigenfunctions are beside their coordinate dependence also functions of the set of quasi-momenta $\mathbf{k} = \{k_j\}_{j=1,\ldots,N+M}$ and the quantities $\mathbf{\Lambda} = \{\Lambda_m\}_{m=1,\ldots,M}$, respectively. The meaning of the latter ones will be discussed below. Since the two sets $\mathbf{x}$ and $\mathbf{y}$ refer to the positions of identical Fermions, we demand $\Psi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda})$ to be completely antisymmetric with respect to the exchange of two coordinates that belong to the same set. Thus the wave function vanishes whenever two particles of the same species come close to each other. A direct consequence of this symmetry is that, since the interaction potential is local, it acts only between the spin-up and the spin-down particles but is invisible for particles of the same species. Therefore terms that correspond to the interaction of two particles of the same sort can be dropped in Eq. (2.2).

The exact eigenfunctions of $H$ can be constructed by means of the Bethe-Ansatz [1]. For the cases $M = 1$ and $M = 2$ they have been found by McGuire [4, 5] and by Flicker and Lieb [50], respectively. The generalization of these results to an arbitrary number $M$ of spin-up particles was not obvious at all. The corresponding problems have been overcome by Yang [6] and Gaudin [7] via the introduction of the nested Bethe-Ansatz hypothesis. The model described by $H$ is therefore also referred to as Yang-Gaudin model.

We briefly comment on how to construct the eigenfunctions of the Hamiltonian (2.2) by using the Bethe-Ansatz. To that end we introduce the set $\mathbf{X} = \{\mathbf{x}, \mathbf{y}\}$, which comprises the coordinates of the spin-up and the spin-down particles. Since we have $N + M$ particles, there exist $(N + M)!$ possible orderings of these. The orderings are referred to as sectors and are usually labeled by the permutations of the integers $1, \ldots, N+M$, that is by the elements $Q \in S(N+M)$ of the symmetric group $S(N+M)$. The ordering of particles which is assigned with the permutation $Q$ reads

$$-\infty \leq X_{Q1} \leq X_{Q2} \leq \cdots \leq X_{Q(N+M)} < +\infty \ . \tag{2.3}$$

For example, let $Q$ be given by $(1, 3, 2, 4, \ldots, N + M)$. The corresponding ordering then reads

$$-\infty \leq X_1 \leq X_3 \leq X_2 \leq X_4 \leq \cdots \leq X_{N+M} \leq +\infty \ . \tag{2.4}$$

The essential idea of the Bethe-Ansatz roots on the following observation: The interaction potential acts only when a spin-up and a spin-down Fermion sit on top of each other, that is at the border between two sectors. Within each sector the Hamiltonian corresponds to the one of free particles and consequently the eigenfunctions are superpositions of plan waves multiplied by amplitudes which are coordinate independent but still might be a function of $\mathbf{k}$ and $\mathbf{\Lambda}$. For the wave function within the sector $Q$ i.e. if the particles are ordered as in Eq. (2.3), this motivates the following Ansatz

$$\Psi(\mathbf{x}, \mathbf{\Lambda}, \mathbf{y}, \mathbf{k})\Big|_Q \propto \sum_{P \in S(N+M)} \sum_{R \in S(M)} [Q|P|R] \exp\left( \imath \sum_{n=1}^{N+M} k_{Pn} X_{Qn} \right) \ , \tag{2.5}$$

where $P$ and $R$ label the permutations of the two sets $k_1, \ldots, k_{N+M}$ and $\Lambda_1, \ldots, \Lambda_M$, respectively. For each of the $(N+M)!$ sectors one makes an Ansatz as in Eq. (2.5). Hence in total there are $((N+M)!)^2 M!$ amplitudes that need to be determined in order to make the wave function explicit. However, not all of these amplitudes are independent. Demanding the wave function to be completely antisymmetric in the sets $\mathbf{k}$ and $\mathbf{\Lambda}$ and taking into account the antisymmetry in $\mathbf{x}$ and $\mathbf{y}$ yields that $N! + 2M! + (N+M)!$ amplitudes are linearly depended. The explicit construction of the amplitudes is a rather messy business. For a detailed derivation we refer to the monograph by Takahashi [29]. Here we sketch the main idea of this procedure.

Denote by $Q$ and $Q'$ the sectors where $x_n$ and $y_m$ are adjacent such that $y_m \leq x_n$ in $Q$ and $x_n \leq y_m$ in $Q'$. The amplitudes within both sectors can be related by application of the matching boundary conditions for the wave function and its first derivative at the borders of $Q$ and $Q'$. The boundary conditions at $x_n = y_m$ read

$$\Psi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda})\Big|_{x_n - y_m = 0^-}^{x_n - y_m = 0^+} = 0 \;, \tag{2.6}$$

$$\left(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial y_m}\right) \Psi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda})\Big|_{x_n - y_m = 0^-}^{x_n - y_m = 0^+} = 4c\Psi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda})\Big|_{x_n = y_m} \;, \tag{2.7}$$

where the condition (2.7) is obtained by integration of the stationary Schrödinger equation in between $x_n - y_m = \mp\epsilon$ and taking the limit $\epsilon \to 0^+$. With the notation $0^\pm$ we indicate that zero is approached from above/below. According to Eqs. (2.6) and (2.7), the wave function is continuous, whereas its derivative has a discontinuity at the points where a spin-up and a spin-down particle are at the same position. Employing the matching boundary conditions to the Bethe-Ansatz wave function (2.5) allows to express the amplitudes within the sector $Q'$ by those of the sector $Q$. Repeating this procedure for all other sectors one finds a set of equations that relates the amplitudes of different sectors with each other. From its solution the amplitudes and thus the Bethe-Ansatz wave function can be determined. Here is how the solution looks like:

Denote by the integer $\tilde{y}_m \in \{1, 2, \ldots, N+M\}$ the location of the spin-up particle with coordinate $y_m$ within the ordering (2.3). For example, if $\tilde{y}_m = 3$ the spin-up particle with coordinate $y_m$ is the third particle from the left and hence $y_m = X_{Q3}$. Let furthermore the positions of the $M$ spin-up particles be ordered such that

$$1 \leq \tilde{y}_1 < \tilde{y}_2 < \cdots < \tilde{y}_M \leq (N+M) \;. \tag{2.8}$$

Then the solution for the amplitudes $[Q|P|R]$ can be cast into the from [6, 7, 29, 31]

$$[Q|P|R] = \mathrm{sgn}(R) \prod_{j<l}^{M} (\Lambda_{Rj} - \Lambda_{Rl} - \imath 2c) \prod_{j=1}^{M} F_P(\tilde{y}_j, \Lambda_{Rj}) \;, \tag{2.9}$$

where

$$F_P(\tilde{y}_j, \Lambda) = \prod_{i=1}^{\tilde{y}_j - 1} (k_{Pi} - \Lambda + \imath c) \prod_{l=\tilde{y}_j + 1}^{N+M} (k_{Pl} - \Lambda - \imath c) \;. \tag{2.10}$$

Even though the Ansatz (2.5) is relatively simple (it is a superposition of plan waves), the entire wave function turns out to be a cumbersome object due to the involved structure of the amplitudes (2.9) and the multiple summations over the symmetric groups in Eq. (2.5). To obtain the full wave function for an arbitrary ordering of particles, Eq. (2.5) has apart from the summation over all permutations $P$ and $R$ to be summed over all permutations $Q$, that is over all possible orderings of particles. Clearly this is a forbiddingly complicated form of the wave function in order to be used in explicit calculations.

The eigenfunctions themselves therefore received little attention and within the traditional approach to treat the system their explicit usage is circumvented. In Sec. 2.3, where the thermodynamic limit is discussed, we briefly sketch the idea of this method.

## 2.2 Reformulation of the eigenfunctions

Although the eigenfunctions can be constructed exactly their complicated and cumbersome structure is little satisfying. Therefore our first aim is to find a more compact form of the wave function (2.5). For the (trivial) case that there is no spin-up particle i.e. $M = 0$, the interaction potential in the Hamiltonian (2.2) drops out. Consequently the system corresponds to the one of free Fermions and the wave function is a Slater-determinant i.e. a determinant of plan waves.

If there are spin-up particles present, the interaction does not drop out. However, following the Bethe-Ansatz the eigenfunctions are still plan waves but with more complicated amplitudes. This suggests that also for the interacting case there exists a determinantal form of the eigenfunctions. Indeed this is the case. We state the reformulation of the exact eigenfunctions as theorem.

**Theorem 1** *Consider the Hamiltonian in Eq.* (2.2)*. Regarding the eigenfunctions of H the following is true:*

1. *The exact eigenfunctions of H can be cast into the form*

$$\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda}) \propto \sum_{R\in S(M)} sgn(R) \prod_{j<l}^{M} [\imath(\Lambda_{Rj}-\Lambda_{Rl}) + 2csgn(y_l - y_j)] \Phi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda}), \tag{2.11}$$

*where $\Phi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})$ is given by the determinant*

$$\Phi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda}) = \det \left[ \left[ \prod_{s=1}^{M} A_j(\Lambda_{Rs}, x_l - y_s) e^{\imath k_j x_l} \right] \Bigg| \left[ \prod_{s\neq m}^{M} A_j(\Lambda_{Rs}, y_m - y_s) e^{\imath k_j y_m} \right] \right]_{\substack{j=1,\ldots,N+M \\ l=1,\ldots,N \\ m=1,\ldots,M}} \tag{2.12}$$

*and*

$$A_j(\Lambda, x) = \imath(k_j - \Lambda) + csgn(x) \ . \tag{2.13}$$

2. *The wave functions* (2.11) *are eigenfunctions of the Hamiltonian* (2.2) *and of the center of mass momentum operator*

$$\mathcal{K} = \frac{1}{\imath}\left(\sum_{n=1}^{N}\frac{\partial}{\partial x_n} + \sum_{m=1}^{M}\frac{\partial}{\partial y_m}\right) \tag{2.14}$$

*to the eigenvalues*

$$E = \sum_{j=1}^{N+M} k_j^2 \quad , \quad K = \sum_{j=1}^{N+M} k_j \tag{2.15}$$

*such that*

$$H\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda}) = E\Psi(\mathbf{x},\mathbf{k},\mathbf{y}\mathbf{\Lambda}) \quad , \quad \mathcal{K}\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda}) = K\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda}) \ . \tag{2.16}$$

The prove of Theorem 1 will be given in Appendix A.1. Here we discuss its implications.

Comparing Eq. (2.11) with the Bethe-Ansatz form (2.5) of the eigenfunctions, reveals that the summation over all sectors in combination with the summation over all permutations of the quasi-momenta has been incorporated into the determinant $\Phi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})$. The full wave function is then expressed as a sum of these. At hand this is a more convenient expression than the original form of the Bethe-Ansatz wave function (2.5), since it allows to employ the power full tools of matrix algebra to manipulate determinants. The main advantage of the expression (2.11) is that the cumbersome summation over all orderings of particles has been carried out. Note that the pre-exponential factors in Eqs. (2.11) and (2.12) are coordinate dependent only via sign-functions and are thus constant within each sector. For a given ordering of particles they resemble the amplitudes in Eq. (2.9).

In Eq. (2.11) we are still left with the summation over all permutations $R \in S(M)$. Hence the reformulated form of the eigenfunctions might be especially useful for the highly imbalanced case, where only a few spin-up particles are present. The case of identical Fermions is restored from Eqs. (2.11) and (2.12) by setting $M = 0$. In this case the last $M$ columns of the determinant (2.12) as well as the summation over the permutations $R \in S(M)$ drop out and the wave function acquires the form of a Slater determinant for non-interacting Fermions.

In Eq. (2.11), the eigenfunctions are determined only up to a coordinate independent constant. For periodic boundary conditions it can be fixed by the normalization condition of the wave function to unity. However, to calculate quantities like expectation values or correlation functions it is not needed.

### 2.2.1 Symmetry properties

We discuss the symmetries of the eigenfunctions (2.11). Due to the determinantal form of $\Phi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})$, the antisymmetry of the wave function when two spin-down particles $x_i \leftrightarrow x_j$ are exchanged is obvious. The same is true regarding the exchange of two quasi-momenta $k_n \leftrightarrow k_l$. To reveal that the wave function (2.11) is antisymmetric with respect

to the exchange of two spin-up particles $y_\mu \leftrightarrow y_\nu$ we observe the following property of $\Phi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda})$ when $y_\mu \leftrightarrow y_\nu$ and simultaneously $\Lambda_{R\mu} \leftrightarrow \Lambda_{R\nu}$ are exchanged

$$\begin{aligned}\Phi(\mathbf{x}, \mathbf{k}, y_1, \ldots, y_\mu, \ldots, y_\nu, \ldots, y_M, \Lambda_{R1}, \ldots, \Lambda_{R\mu}, \ldots, \Lambda_{R\nu}, \ldots, \Lambda_{RM}) \\ = -\Phi(\mathbf{x}, \mathbf{k}, y_1, \ldots, y_\nu, \ldots, y_\mu, \ldots, y_M, \Lambda_{R1}, \ldots, \Lambda_{R\nu}, \ldots, \Lambda_{R\mu}, \ldots, \Lambda_{RM}).\end{aligned} \tag{2.17}$$

To show the antisymmetry of the full wave function when exchanging $y_\mu \leftrightarrow y_\nu$ we write Eq. (2.11) as

$$\Psi(\mathbf{x}, \mathbf{k}, y_1, \ldots, y_\nu, \ldots, y_\mu, \ldots, y_M, \mathbf{\Lambda}) = \sum_{R \in S(M)} \operatorname{sgn}(R) \tag{2.18}$$

$$\prod_{\substack{j<l \\ \neq \mu,\nu}}^{M} [\imath(\Lambda_{Rj} - \Lambda_{Rl}) + 2c\operatorname{sgn}(y_l - y_j)] \Big|_{y_\mu \leftrightarrow y_\nu} \tag{2.19}$$

$$\prod_{j<\mu} [\imath(\Lambda_{Rj} - \Lambda_{R\mu}) + 2c\operatorname{sgn}(y_\mu - y_j)] \prod_{\substack{j<\nu \\ \neq \mu}} [\imath(\Lambda_{Rj} - \Lambda_{R\nu}) + 2c\operatorname{sgn}(y_\nu - y_j)] \Big|_{y_\mu \leftrightarrow y_\nu} \tag{2.20}$$

$$\prod_{\substack{l>\mu \\ \neq \nu}}^{M} [\imath(\Lambda_{R\mu} - \Lambda_{Rl}) + 2c\operatorname{sgn}(y_l - y_\mu)] \prod_{l>\nu}^{M} [\imath(\Lambda_{R\nu} - \Lambda_{Rl}) + 2c\operatorname{sgn}(y_l - y_\nu)] \Big|_{y_\mu \leftrightarrow y_\nu} \tag{2.21}$$

$$[\imath(\Lambda_{R\mu} - \Lambda_{R\nu}) + 2c\operatorname{sgn}(y_\nu - y_\mu)] \Big|_{y_\mu \leftrightarrow y_\nu} \Phi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda}) \Big|_{y_\mu \leftrightarrow y_\nu} . \tag{2.22}$$

Using the property (2.17) reveals that the terms in lines (2.19)-(2.22) remain invariant if simultaneously $\Lambda_{R\mu} \leftrightarrow \Lambda_{R\nu}$ are exchanged. However, doing so we get a minus from the sign-function $\operatorname{sgn}(R)$ on the right hand side of line (2.18). Thus we have

$$\Psi(\mathbf{x}, \mathbf{k}, y_1, \ldots, y_\nu, \ldots, y_\mu, \ldots, y_M, \mathbf{\Lambda}) = -\Psi(\mathbf{x}, \mathbf{k}, y_1, \ldots, y_\mu, \ldots, y_\nu, \ldots, y_M, \mathbf{\Lambda}) \ . \tag{2.23}$$

To reveal the antisymmetry of Eq. (2.11) when exchanging $\Lambda_j \leftrightarrow \Lambda_l$, we write the wave function as in the equation above. The difference is that now the indices $\mu$ and $\nu$ have to be chosen for each permutation $R$ on the right hand side of Eq. (2.18) such that $\Lambda_{R\mu} = \Lambda_j$ and $\Lambda_{R\nu} = \Lambda_l$. Then exchanging $\Lambda_j \leftrightarrow \Lambda_l$ and simultaneously $y_\mu \leftrightarrow y_\nu$ we can again make use of Eq. (2.17) in order to deduce

$$\begin{aligned}\Psi(\mathbf{x}, \mathbf{k}, y_1, \ldots, y_\mu, \ldots, y_\nu, \ldots, y_M, \Lambda_1, \ldots, \Lambda_j, \ldots, \Lambda_l, \ldots, \Lambda_M) \\ = \Psi(\mathbf{x}, \mathbf{k}, y_1, \ldots, y_\nu, \ldots, y_\mu, \ldots, y_M, \Lambda_1, \ldots, \Lambda_l, \ldots, \Lambda_j, \ldots, \Lambda_M)\end{aligned} \tag{2.24}$$

which together with Eq. (2.23) shows

$$\Psi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \Lambda_1, \ldots, \Lambda_l, \ldots, \Lambda_j, \ldots, \Lambda_M) = -\Psi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \Lambda_1, \ldots, \Lambda_j, \ldots, \Lambda_l, \ldots, \Lambda_M) \tag{2.25}$$

the antisymmetry with respect to the exchange of $\Lambda_j \leftrightarrow \Lambda_l$.

We emphasize, that the wave function (2.11) has no well defined symmetry when a spin-up and a spin-down particle are exchanged. This reflects the fact that the two particle species are distinguishable.

## 2.3 Bethe-Ansatz equations and thermodynamic limit

Employing periodic boundary conditions, we derive in Sec. 2.3.1 the Bethe-Ansatz equations. Their solution for the ground state is the topic of Sec. 2.3.2. The thermodynamic limit is discussed in Sec. 2.3.3.

### 2.3.1 Periodic boundary conditions

In order to acquire results for finite particle density, we subject the wave function (2.11) to periodic boundary conditions. This yields quantization rules for the quasi-momenta $\{k_j\}_{j=1,\ldots,N+M}$ and the quantities $\{\Lambda_l\}_{l=1,\ldots,M}$ in form of transcendental equations which are known as Bethe-Ansatz equations.

Demanding the eigenfunctions to be periodic with period $L$, the corresponding conditions read

$$\Psi(x_1,\ldots,x_\mu+L,\ldots,x_N,\mathbf{y},\mathbf{k},\mathbf{\Lambda}) = \Psi(x_1,\ldots,x_\mu,\ldots,x_N,\mathbf{y},\mathbf{k},\mathbf{\Lambda}) \ , \tag{2.26}$$

$$\Psi(\mathbf{x},y_1,\ldots,y_\nu+L,\ldots,y_M,\mathbf{k},\mathbf{\Lambda}) = \Psi(\mathbf{x},y_1,\ldots,y_\nu,\ldots,y_M,\mathbf{k},\mathbf{\Lambda}) \ , \tag{2.27}$$

where $\mu = 1,\ldots,N$ and $\nu = 1,\ldots,M$. We consider the condition (2.26). Since the coordinate $x_\mu$ only appears in the $\mu$-th column of the determinant $\Phi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})$, see Eq. (2.12), the corresponding entries on the left and on the right hand side of Eq. (2.26) can directly be compared. This yields

$$e^{\imath k_j L} = \prod_{m=1}^{M} e^{-2\imath \arctan((k_j-\Lambda_m)/c)} \ , \tag{2.28}$$

where we have used that the right hand side of the equation above is invariant under permutations of the set $\mathbf{\Lambda}$. Taking the logarithm of Eq. (2.28) we obtain

$$k_j L = 2\pi n_j - 2\sum_{m=1}^{M} \arctan\left(\frac{k_j-\Lambda_m}{c}\right) \quad , \quad j = 1,\ldots,N+M \ . \tag{2.29}$$

Here the branch of $\arctan(x)$ is chosen such that $-\pi/2 \leq \arctan(x) \leq \pi/2$. The quantum numbers $n_j$ originate from the ambiguity of the phase modulo $2\pi$ and are either integers for $M$ even or half-odd integers for $M$ odd.

We turn to the periodicity condition with respect to the spin-up particles. To evaluate it, we decompose the left and right hand side of Eq. (2.27) into terms which depend on $y_\nu$ and those which are independent. Then dividing out all factors that remain invariant under the shift $y_\nu \to y_\nu + L$, this condition acquires the form

$$e^{\imath k_\nu L} = \prod_{\substack{s=1\\ \neq\nu}}^{M} e^{-2\imath \arctan((k_\nu-\Lambda_s)/c)} \prod_{\substack{j=1\\ \neq\nu}}^{N+M} e^{2\imath \arctan((k_j-\Lambda_\nu)/c)} \prod_{\substack{l=1\\ \neq\nu}}^{M} e^{-2\imath \arctan((\Lambda_l-\Lambda_\nu)/(2c))} \ , \tag{2.30}$$

where we have used that Eq. (2.27) has to hold for an arbitrary permutation of the quantities $\mathbf{\Lambda}$. Taking the logarithm yields

$$k_\nu L = 2\pi\widetilde{m}_\nu - 2\sum_{\substack{s=1\\ \neq\nu}}^{M} \arctan\left(\frac{k_\nu - \Lambda_s}{c}\right) + 2\sum_{\substack{j=1\\ \neq\nu}}^{N+M} \arctan\left(\frac{k_j - \Lambda_\nu}{c}\right) \\ -2\sum_{\substack{j=1\\ \neq\nu}}^{M} \arctan\left(\frac{\Lambda_j - \Lambda_\nu}{2c}\right) \quad , \quad \nu = 1, \ldots, N+M \ . \tag{2.31}$$

Here the quantities $\widetilde{m}_j$ are integers for $N+M$ odd and half-odd integers for $N+M$ even. Combining Eqs. (2.29) and (2.31) yields the Bethe-Ansatz equations [6, 7, 29]

$$k_j L = 2\pi n_j - 2\sum_{m=1}^{M} \arctan\left(\frac{k_j - \Lambda_m}{c}\right) \quad , \quad j = 1, \ldots, N+M \ , \tag{2.32}$$

$$\pi J_\mu = \sum_{j=1}^{N+M} \arctan\left(\frac{k_j - \Lambda_\mu}{c}\right) + \sum_{\substack{\nu=1\\ \neq\mu}}^{M} \arctan\left(\frac{\Lambda_\mu - \Lambda_\nu}{2c}\right) \quad , \quad \mu = 1, \ldots, M \ , \tag{2.33}$$

where the quantum numbers $J_\mu$ are integers for $N$ odd and half-odd integers for $N$ even. These are $N+2M$ coupled algebraic equations for the $N+M$ quasi-momenta $\{k_j\}_{j=1,\ldots,N+M}$ and the $M$ quantities $\{\Lambda_l\}_{l=1,\ldots,M}$. The two sets of quantum numbers $\{n_j\}_{j=1,\ldots,N+M}$ and $\{J_\mu\}_{\mu=1,\ldots,M}$ determine the solutions for $\mathbf{k}$ and $\mathbf{\Lambda}$ and thus characterize the state of the system. In particular all quantum numbers $n_j$ have to be distinct, since otherwise two or more quasi-momenta coincide and the wave function vanishes identically. Likewise the antisymmetry of the wave function in $\mathbf{\Lambda}$ implies that all quantum numbers $J_\mu$ have to be distinct as well. For convenience we will always assume $N$ to be odd and $M \leq N$. Finally we remark that the spectrum of the quantum numbers $J_\mu$ is according to Eq. (2.33) bounded by

$$-\frac{N+2M-1}{2} \leq J_\mu \leq \frac{N+2M-1}{2} \ . \tag{2.34}$$

Consequently there are at most $N+2M$ allowed values for $J_\mu$.

### 2.3.2 Solution of the Bethe-Ansatz equations and ground state

We discuss how the quantum numbers have to be chosen in order to obtain the ground state i.e. the state with lowest energy. To that end we consider the limit of vanishing interaction strength. The cases of odd and even $M$ have to be considered separately.

For $M$ odd, the quasi-momenta of the non-interacting ground state are

$$\mathbf{k} = \frac{2\pi}{L}\left\{0^{\pm}, \pm 1^{\pm}, \ldots, \pm\left(\frac{M-1}{2}\right)^{\pm}, \ldots, \pm\frac{N-1}{2}\right\} \ , \tag{2.35}$$

where the superscript indicates if the corresponding quasi-momentum for $c = 0$ is approached from above or below from the one for $c > 0$ and shows that the state is doubly occupied for $c = 0$. Since the quantum numbers which determine the ground state are the same for $c = 0$ and $c > 0$, they can be extracted from the Bethe-Ansatz equations (2.32) and (2.33) by demanding that these yield in the limit $c \to 0^+$ the set (2.35). Following the discussion after Eq. (2.29), the quantum numbers $n_j$ are half-odd integers for $M$ odd. For convenience we write them as $n_j = m_j + 1/2$, where the $m_j$'s are integers. Making the Ansatz

$$\{m_j\}_{j=1,\ldots,N+M} = \left\{0, \pm 1, \ldots, \pm\frac{N+M-2}{2}, -\frac{N+M}{2}\right\}, \tag{2.36}$$

$$\{J_\mu\}_{\mu=1,\ldots,M} = \left\{0, \pm 1, \ldots, \pm\frac{M-1}{2}\right\}, \tag{2.37}$$

we substitute Eqs. (2.35)-(2.37) into the Bethe-Ansatz equations. One can verify by a direct calculation that these become consistent, if simultaneously the set $\mathbf{\Lambda}$ is chosen as

$$\mathbf{\Lambda} = \frac{2\pi}{L}\left\{0, \pm 1, \ldots, \pm\frac{M-1}{2}\right\}. \tag{2.38}$$

Consequently the quantum numbers for the ground state are determined by Eqs. (2.36) and (2.37) and the corresponding solution to the Bethe-Ansatz equations is for vanishing interaction strength given by the sets (2.35) and (2.38).

For $M$ even, the quasi-momenta of the non-interacting ground state are

$$\mathbf{k} = \frac{2\pi}{L}\left\{0^{\pm}, \pm 1^{\pm}, \ldots, \pm\left(\frac{M}{2}-1\right)^{\pm}, +\left(\frac{M}{2}\right)^{\pm}, -\frac{M}{2}, \ldots, \pm\frac{N-1}{2}\right\}. \tag{2.39}$$

As for $M$ odd, one verifies by substituting the set (2.39) as well as

$$\{n_j\}_{j=1,\ldots,N+M} = \left\{0, \pm 1, \ldots, \pm\frac{N+M-1}{2}\right\}, \tag{2.40}$$

$$\{J_\mu\}_{\mu=1,\ldots,M} = \left\{0, \pm 1, \ldots, \pm\left(\frac{M}{2}-1\right), \frac{M}{2}\right\}, \tag{2.41}$$

$$\mathbf{\Lambda} = \frac{2\pi}{L}\left\{0, \pm 1, \ldots, \pm\left(\frac{M}{2}-1\right), \frac{M}{2}\right\} \tag{2.42}$$

into the Bethe-Ansatz equations that these are consistently fulfilled. The quantum numbers for the ground state and $M$ even, are therefore given by Eqs. (2.40) and (2.41) and the corresponding solutions for $\mathbf{k}$ and $\mathbf{\Lambda}$ are in the non-interacting limit determined by Eqs. (2.39) and (2.42).

The solution of the Bethe-Ansatz equations for vanishing interaction strength shows that the quantities $\Lambda_1, \ldots, \Lambda_M$ coincide with the quasi-momenta of the single particle states that become for $c \to 0^+$ doubly occupied. Thus in the limit of vanishing interaction strength, $\Lambda_1, \ldots, \Lambda_M$ might be interpreted as the quasi-momenta of the spin-up particles and $J_1, \ldots, J_M$ as the corresponding quantum numbers.

For $c \to \infty$ we obtain from the Bethe-Ansatz equations

$$k_j = \frac{2\pi}{L}\left(n_j + \frac{1}{\pi}\sum_{\mu=1}^{M}\arctan(\lambda_\mu)\right) , \tag{2.43}$$

$$\pi J_\mu = -(N+M)\arctan(\lambda_\mu) + \sum_{\nu\neq\mu}^{M}\arctan\left(\frac{\lambda_\mu - \lambda_\nu}{2}\right) . \tag{2.44}$$

Here we have introduced the quantities $\lambda_\mu = \Lambda_\mu/c$, which take into account that $\Lambda_\mu$ scales like $c$ for $c \to \infty$, see Ref. [29].

For finite $c$ the Bethe-Ansatz equations need to be calculated numerically. As $c$ increases the quasi-momenta evolve smoothly from multiple integer values of $2\pi/L$ to multiple integer values of $2\pi/L$ with an offset that according to Eq. (2.43) is determined by all $\lambda_\mu$'s. The parameter $\Lambda_\mu$ evolves smoothly from its initial value to $\pm\infty$ as the interaction increases. Figure 2.1 schematically depicts the evolution of the quasi-momenta for the ground state and $M$ odd as $c$ varies from $c = \infty$ to $c = 0$.

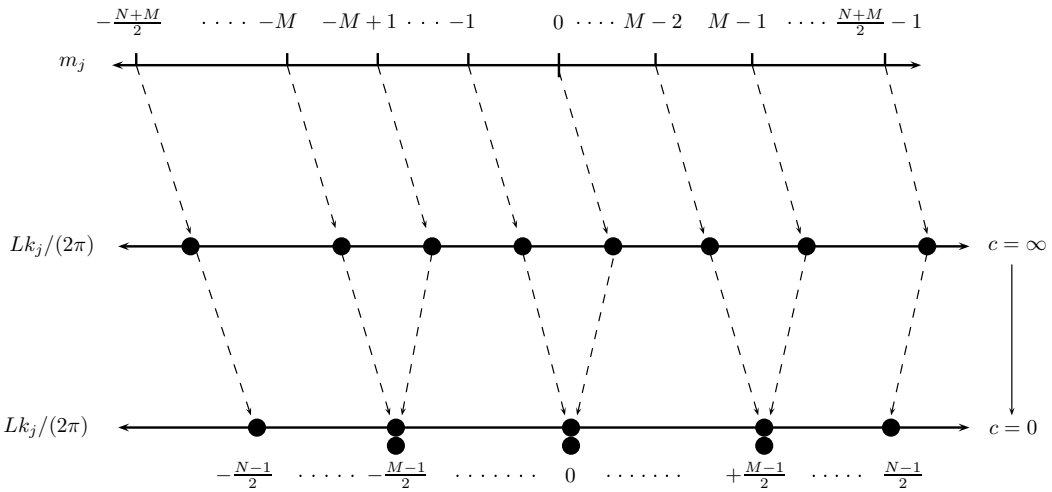


Figure 2.1: Schematic picture for the evolution of the quasi-momenta for the ground state as $c$ varies from $\infty$ to zero with $M$ odd. The quantum numbers are determined by Eqs. (2.36) and (2.37). The upper line represents the set of quantum numbers $m_j$ as given (2.36). For $c = \infty$ the values for $Lk_j/(2\pi)$ have an offset of $+1/2$ with respect to the quantum numbers $m_j$ (middle line). The lower line corresponds to the values of $Lk_j/(2\pi)$ in the limit $c \to 0^+$. The arrows indicate the evolution of the quasi-momenta as $c$ varies and show the quantum numbers that yield to doubly occupied single particle states for $c = 0$.

For the systems overall momentum $K$, the Bethe-Ansatz equations (2.32) and (2.33)

yield

$$K = \sum_{j=1}^{N+M} k_j = \frac{2\pi}{L} \left( \sum_{j=1}^{N+M} n_j - \sum_{l=1}^{M} J_l \right) \ . \tag{2.45}$$

For the ground state the quantum numbers $n_j$ are given by Eqs. (2.36) and (2.40). Substituting them into the equation above revels that the first sum vanishes such that the sum over the quantum numbers $J_\mu$ is directly related to the overall momentum.

### 2.3.3 Thermodynamic limit

In the thermodynamic limit, where the size $L$ of the system tends to infinity, there are different possibilities for the scaling of the particle numbers.

In the balanced case, the numbers of the spin-up and the spin-down particles tend to infinity such that both, the density of the spin-up Fermions $M/L$ as well as the density of the spin-down Fermions $N/L$, remain finite. This case has been studied extensively. A comprehensive review can be found in Takahashi's book [29] or likewise in Ref. [31].

We briefly comment on the approach how this case is treated. For the ground state, the quasi-momenta $\mathbf{k}$ and the quantities $\mathbf{\Lambda}$ are assumed to be distributed with the symmetric densities $\varrho(k) = \varrho(-k)$ and $\sigma(\Lambda) = \sigma(-\Lambda)$ in between the bounds $\pm q$ and $\pm Q$, respectively. The normalization conditions

$$N + M = \int_{-q}^{+q} dk \varrho(k) \qquad , \qquad M = \int_{-Q}^{+Q} d\Lambda \sigma(\Lambda) \tag{2.46}$$

determine the bounds $q$ and $Q$. The density distribution functions themselves are determined via the Bethe-Ansatz equations, which in the thermodynamic limit can be shown to be equivalent to a set of coupled integral equations for $\varrho(k)$ and $\sigma(\Lambda)$ [6, 7, 31]

$$\varrho(k) = \frac{1}{2\pi} + \frac{c}{\pi} \int_{-Q}^{+Q} d\Lambda \frac{\sigma(\Lambda)}{(k-\Lambda)^2 + c^2} \ , \tag{2.47}$$

$$\sigma(\Lambda) = \frac{c}{\pi} \int_{-q}^{+q} dk \frac{\varrho(k)}{(k-\Lambda)^2 + c^2} - \frac{2c}{\pi} \int_{-Q}^{+Q} d\Lambda' \frac{\sigma(\Lambda')}{(\Lambda-\Lambda')^2 + 4c^2} \ . \tag{2.48}$$

Based on their solution for $\varrho(k)$ and $\sigma(\Lambda)$, the zero temperature thermodynamics i.e. the thermodynamics of the ground state can be calculated. Although the treatment sketched above has successfully been extended to finite temperature [60, 68, 69], the approach suffers from the drawback that an analytical solution of the integral equations (2.47) and (2.48) and hence explicit expressions for the ground state properties, are available only in some limiting cases [29].

In the highly imbalanced case, the thermodynamic limit is taken only for one of the particle species. From an analytical point of view this scenario is more favorable, since the above mentioned problem regarding the density distribution function drops out and an explicit expression for $\varrho(k)$ can be obtained. Moreover, as has been emphasized in Sec. 2.2, this is exactly the situation where the reformulated eigenfunctions acquire a particular handy form. We consider the case where the thermodynamic limit is taken only for the spin-down particles. While $N, L \to \infty$ such that the particle density $N/L$ remains finite, the density of the spin-up Fermions vanishes in the thermodynamic limit. We refer to the spin-up and the spin-down particles also as minority and majority Fermions, respectively.

For a dense set of quantum numbers as in Eqs. (2.36) and (2.40), the density of states $\varrho(k) = \partial n(k)/\partial k$ can be obtained by taking the derivative of the first Bethe-Ansatz equation (2.32). This yields

$$\varrho(k) = \frac{L}{2\pi} + \sum_{l=1}^{M} \frac{1}{\pi} \frac{c}{(k-\Lambda_l)^2 + c^2} \ . \tag{2.49}$$

In leading order of $L$ this corresponds to the momentum distribution of a sea of free Fermions. The quasi-momenta distribute themselves with the constant density $L/(2\pi)$ between two values $k_\pm$. The $M$ additional terms on the right hand side of Eq. (2.49) then might be interpreted as momentum distribution for the spin-up particles. Each of them has the form of a Lorentzian-distribution centered around $\Lambda_l$. For $c \to 0^+$ they yield $\delta$-functions at $k = \Lambda_l$.

The momenta $k_\pm$ are defined through the normalization condition of the density of states and the normalization of its first moment to the overall momentum of the system

$$\begin{aligned} N + M &\stackrel{!}{=} \int_{k_-}^{k_+} dk \ \varrho(k) \ , \\ K &\stackrel{!}{=} \int_{k_-}^{k_+} dk \ k\varrho(k) \ . \end{aligned} \tag{2.50}$$

These are two transcendental equations for $k_+$ and $k_-$. Assuming that the non-interacting Fermi-sea is at rest, the solutions can be expanded in inverse powers of the system size $L$. We obtain

$$k_\pm = \pm\frac{\pi N}{L} \pm \frac{\pi}{L} \sum_{\mu=1}^{M} \left[ 1 - \frac{1}{\pi} \arctan\left( \frac{k_\mathrm{F} \mp \Lambda_\mu}{c} \right) \right] + \mathcal{O}(L^{-2}) \ . \tag{2.51}$$

Therefore in leading order of $L$, the momentum distribution is symmetric i.e. $k_+ = -k_-$ and $k_+$ coincides with the Fermi-momentum $k_\mathrm{F}$ of $N$ non-interacting Fermions i.e. $k_+ = k_\mathrm{F} = \pi N/L$. Nevertheless, the correction terms to the non-interacting Fermi-momentum in Eq. (2.51) are essential when discussing the energy shift due to the interaction.

In the thermodynamic limit there are different choices for the scaling of the interaction strength

$$c \;=\; \begin{cases} \bar{c}L^{-1} & \\ \hat{c}\;k_{\mathrm{F}} & \\ \infty \qquad \text{(hardcore limit)} & \end{cases} \quad . \tag{2.52}$$

The Lorentzian part in Eq. (2.49) is scaled out by the part due to the Fermi-sea in all three cases.

Using Eq. (2.49), the second Bethe-Ansatz equation (2.33) can be written as

$$K_\mu \;=\; -u(\Lambda_\mu) - w_1(\Lambda_\mu) - \sum_{\substack{\nu=1 \\ \neq\mu}}^{M} w_2(\Lambda_\mu,\Lambda_\nu)\;. \tag{2.53}$$

Here we have introduced the quantities $K_\mu = -2\pi J_\mu/L$. They have the dimension of a momentum and vary in the thermodynamic limit in between $\pm k_{\mathrm{F}}$ as $J_\mu$ varies in between the bounds given in Eq. (2.34). Furthermore we have introduced in Eq. (2.53) the notations

$$u(\Lambda) = \frac{1}{\pi}\int_{-k_{\mathrm{F}}}^{+k_{\mathrm{F}}} dk \arctan\left(\frac{k-\Lambda}{c}\right) \qquad , \qquad w_1(\Lambda) = \frac{\pi}{L}\frac{\partial u(\Lambda)}{\partial k_{\mathrm{F}}} \tag{2.54}$$

$$w_2(\Lambda_\mu,\Lambda_\nu) = \frac{2}{L}\arctan\left(\frac{\Lambda_\mu-\Lambda_\nu}{2c}\right) + \frac{2}{L\pi}\int_{-k_{\mathrm{F}}}^{+k_{\mathrm{F}}} dk \frac{c}{(k-\Lambda_\nu)^2+c^2}\arctan\left(\frac{k-\Lambda_\mu}{c}\right)\;.$$

The terms comprised by $w_2$ couple $\Lambda_\mu$ and $\Lambda_\nu$. However, both $w_1$ and $w_2$, are of the order $1/L$ and hence vanish in the infinite system. Thus in the thermodynamic limit $\Lambda_\mu$ is determined by $K_\mu$ only. For $c > 0$ the function $u(\Lambda)$ in (2.54) is monotonously decreasing in $\Lambda$. Thus it can be inverted. To this end the identity $\arctan(x) = \mathrm{sgn}(x)\pi/2 - \arctan(x^{-1})$ is substituted into equation (2.54). The first term containing the sign-function can be integrated and one obtains an implicit equation for $\Lambda_\mu$. Iterating this equation, we can express $\Lambda_\mu$ up to the first order in $c$ as function of $k_{\mathrm{F}}, K_\mu$ and $c$

$$\Lambda_\mu \;\approx\; K_\mu + \frac{c}{\pi}\ln\left|\frac{k_{\mathrm{F}}+K_\mu}{k_{\mathrm{F}}-K_\mu}\right| + \mathcal{O}(c^2)\;. \tag{2.55}$$

From this equation follows in particular that $\Lambda_\mu \to \mp\infty$ as $K_\mu \to \pm k_{\mathrm{F}}$.

## 2.4 Hubbard Model

We consider the discretized counterpart of the model discussed in the forgoing sections, that is the one-dimensional Hubbard Model. The purpose of this section is to show that the reformulation of the exact eigenfunctions, found for the continuous case, might carry

over to the lattice model. Therefore we will try to keep formulas at a minimum and whenever possible refer ourselves to the continuous case.

The Hubbard model, named after J. Hubbard who proposed the model in the early sixties, was introduced with the aim to describe electrons in narrow energy bands [70]. Nowadays it is a standard model of condensed matter physics. The one-dimensional Hubbard model was proven to be exactly solvable by Lieb and Wu [71], which based on the earlier works [6, 7] of Yang and Gaudin for the continuous model, constructed the exact eigenstates via the nested Bethe-Ansatz. The state of the art of the subject has been summarized in the recently published monograph [47].

The setup of the one-dimensional Hubbard model is as follows: We consider spin one-half Fermions in a one-dimensional equidistant lattice with lattice constant $a \geq 0$. The lattice is assumed to be made up of $\tilde{L}$ sites which are labeled by the integers $l = 1, \ldots, \tilde{L}$. In second quantization the Hubbard Hamiltonian is written as

$$\mathcal{H} = -t \sum_{<i,j>} \sum_{\sigma \in \{\uparrow\downarrow\}} \left( c^{\dagger}_{i\sigma} c_{j\sigma} + c^{\dagger}_{j\sigma} c_{i\sigma} \right) + U \sum_{j=1}^{\tilde{L}} c^{\dagger}_{j\uparrow} c_{j\uparrow} c^{\dagger}_{j\downarrow} c_{j\downarrow} \ , \tag{2.56}$$

where the operators $c^{(\dagger)}_{i\sigma}$ annihilate (create) a Fermion with spin $\sigma$ at the lattice site $i$. They fulfill the fermionic anticommutator relations

$$\{c_{i\sigma}, c_{j\sigma'}\} = \{c^{\dagger}_{i\sigma}, c^{\dagger}_{j\sigma'}\} = 0 \qquad \{c^{\dagger}_{i\sigma}, c_{j\sigma'}\} = \delta_{ij}\delta_{\sigma\sigma'}. \tag{2.57}$$

The notation $\langle i, j \rangle$ in Eq. (2.56) indicates that the summation over the lattice sites $i, j$ is restricted to nearest neighbors i.e. $i = j \pm 1$. We briefly comment on the terms in Eq. (2.56): The first term in the Hamiltonian $\mathcal{H}$ represents the hopping of particles in between nearest neighbor sites. This process is associated with the energy $t > 0$. The second term accounts for the interaction. Since it involves only creation and annihilation operators at the same lattice site, the interaction is local and hence acts only between particles with different spin (due to Pauli's principal two Fermions with the same spin can not be simultaneously at the same lattice site). The interaction strength $U > 0$ is also referred to as on-site energy, since it corresponds to the amount of energy needed to have two particles with opposite spin at the same lattice site.

The vacuum state $|0\rangle$ is defined by

$$c_{l\sigma}|0\rangle \equiv 0 \quad \text{for arbitrary} \quad l = 1, \ldots, \tilde{L} \quad , \quad \sigma = \{\uparrow, \downarrow\} \tag{2.58}$$

and corresponds to the empty lattice. By successive application of the creation operator $c^{\dagger}_{i\sigma}$ the $N + M$ particle state, where $N$ spin-up and $M$ spin-down particles are present, is obtained from the vacuum by

$$|\varphi_{N,M}\rangle = \sum_{x'_1, \ldots, x'_N}^{\tilde{L}} \sum_{y'_1, \ldots, y'_M}^{\tilde{L}} \varphi(\mathbf{x}', \mathbf{k}, \mathbf{y}', \mathbf{\Lambda}) \prod_{j=1}^{N} c^{\dagger}_{x'_j\downarrow} \prod_{l=1}^{M} c^{\dagger}_{y'_l\uparrow} |0\rangle \ . \tag{2.59}$$

Here the elements of the sets $\mathbf{x}' = \{x'_1, \ldots, x'_N\}$ and $\mathbf{y}' = \{y'_1, \ldots, y'_M\}$ can take integer values of the lattice constant i.e. the values $a, 2a, \ldots, a\tilde{L}$. They label the positions of the

spin-down and the spin-up particles, respectively. The function $\varphi(\mathbf{x}', \mathbf{k}, \mathbf{y}', \mathbf{\Lambda})$ is what, in analogy to the continuous case, might be called the wave function in coordinate representation. As there, it is characterized by the two sets $\mathbf{k} = \{k_j\}_{j=1,\ldots,N+M}$ and $\mathbf{\Lambda} = \{\Lambda_l\}_{l=1,\ldots,M}$. From Eq. (2.59) it can be obtained by projecting $|\varphi_{N,M}\rangle$ onto the $N+M$ particle state $|\mathbf{x}, \mathbf{y}\rangle = \prod_{j=1}^{N} c^{\dagger}_{x_j\downarrow} \prod_{l=1}^{M} c^{\dagger}_{y_l\uparrow}|0\rangle$, that is

$$\varphi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda}) = \langle \mathbf{x}, \mathbf{y} | \varphi_{N,M} \rangle \ . \tag{2.60}$$

Similarly the action of the Hamiltonian in first quantization onto $\varphi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda})$ is obtained by acting with $\mathcal{H}$ on (2.59) and projection onto the $N+M$ particle state $|\mathbf{x}, \mathbf{y}\rangle$. For the first quantized version of the stationary Schrödinger equation

$$\langle \mathbf{x}, \mathbf{y} | \mathcal{H} | \varphi_{N,M} \rangle = E \langle \mathbf{x}, \mathbf{y} | \varphi_{N,M} \rangle \tag{2.61}$$

this yields

$$-t \sum_{s=\pm a} \left( \sum_{j=1}^{N} \varphi(x_1, \ldots, x_j + s, \ldots, x_N, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda}) + \sum_{l=1}^{M} \varphi(\mathbf{x}, \mathbf{k}, y_1, \ldots, y_l + s, \ldots, y_M, \mathbf{\Lambda}) \right)$$
$$+U \sum_{j=1}^{N} \sum_{l=1}^{M} \delta_{x_j y_l} \varphi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda}) = E \varphi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda}) \ . \tag{2.62}$$

The equation above is a *difference* equation for the function $\varphi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda})$. It is the discretized counterpart of the stationary Schrödinger equation (2.2). Analogously to the continuous case it can be solved by means of the Bethe-Ansatz. The construction of the exact eigenfunctions follows along the same lines as described in Sec. 2.1. A detailed derivation can be found in Refs. [29, 47].

We introduce the set $\mathbf{X} = \{\mathbf{x}, \mathbf{y}\}$, which comprises the lattice coordinates of all particles. The sector where the particles are ordered according to

$$a \leq X_{Q1} \leq \ldots \leq X_{Q(N+M)} \leq a\tilde{L} \ , \tag{2.63}$$

is labeled by the permutation $Q \in S(N+M)$. Within the sector $Q$ the Bethe-Ansatz for the eigenfunctions reads

$$\varphi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda})\Big|_Q \propto \sum_{P \in S(N+M)} \sum_{R \in S(M)} [Q|P|R] \exp\left( \imath \sum_{n=1}^{N+M} k_{Pn} X_{Qn} \right) \ . \tag{2.64}$$

The amplitudes can be determined by a similar procedure to the one described in the continuous case. To present their solution we introduce in analogy to the discussion there the quantities $\tilde{y}_m$ which label the position of the spin-up particle with lattice coordinate $y_m$ within the ordering (2.63). For the ordering of spin-up particles such that $1 \leq \tilde{y}_1 < \ldots < \tilde{y}_M \leq N+M$ the coefficients in the equation above can be written as [71, 29]

$$[Q|P|R] = \mathrm{sgn}(R) \prod_{j<l}^{M} \left( \Lambda_{Rj} - \Lambda_{Rl} - \imath \frac{U}{2t} \right) \prod_{j=1}^{M} F_P(\tilde{y}_j, \Lambda_{Rj}) \ , \tag{2.65}$$

where

$$F_P(\tilde{y}_j,\Lambda)=\prod_{i=1}^{\tilde{y}_j-1}\left(\frac{\sin(ak_{Pi})}{a}-\Lambda+\imath\frac{U}{4t}\right)\prod_{l=\tilde{y}_j+1}^{N+M}\left(\frac{\sin(ak_{Pl})}{a}-\Lambda-\imath\frac{U}{4t}\right)\ . \tag{2.66}$$

Hence the amplitudes for the Hubbard model resemble those for the continuous case. Indeed, comparing Eqs. (2.65) and (2.9), it is found that both expressions become identical if we identify

$$k_j\longleftrightarrow\frac{\sin(k_ja)}{a}\qquad,\qquad c\longleftrightarrow\frac{U}{4t}\ . \tag{2.67}$$

This immediately leads to the conjecture that the eigenfunctions of the Hubbard model can be cast into a similar form as the eigenfunctions of the continuous model in Eq. (2.11).

We demonstrate that for the simplest non-trivial case, where $N$ spin-down but only a single spin-up particle is present. Following the observation above, we assume the eigenfunctions to be of the form

$$\varphi(\mathbf{x},\mathbf{k},y,\Lambda)\propto\det\left[\left(\imath(\sin(k_ja)/a-\Lambda)+\frac{U}{4t}\mathrm{sgn}(x_l-y)\right)e^{\imath k_jx_l}\,\middle|\,e^{\imath k_jy}\right]_{\substack{j=1,\ldots,N+1\\ l=1,\ldots N}}. \tag{2.68}$$

Substituting this into the stationary Schrödinger equation reveals that $\varphi(\mathbf{x},\mathbf{k},y,\Lambda)$ is indeed a solution of Eq. (2.62) to the eigenvalue

$$E=-2t\sum_{j=1}^{N+1}\cos(ak_j)\ . \tag{2.69}$$

We proof this statement in Sec. A.2 of Appendix A.

In the continuous limit, where the number of lattice sites $\tilde{L}\to\infty$ and simultaneously the lattice constant $a\to 0$ such that the product $\tilde{L}a=L$ remains finite, the coordinates $\mathbf{x},\mathbf{y}$ can be treated as the particle positions on the interval $[0,L]$. As is readily seen Eq. (2.68) becomes identical with the eigenfunctions of the continuous case; compare Eq. (2.11) for $M=1$.

This result might be generalized to an arbitrary number $M$ of spin-up particles. However, in the following we exclusively study the continuous case. Therefore we stop the discussion of the Hubbard model at this point and turn to applications of the results obtained during this chapter.

# Chapter 3

# Results for one spin-up Fermion

In this chapter, we consider McGuire's model [4, 5] i.e. a single minority particle that interacts via repulsive contact interaction with an arbitrary number of identical Fermions. This corresponds to the arbitrary $N$ and $M = 1$ case of the Yang-Gaudin model introduced in the forgoing chapter. The determinantal form of the exact eigenfunctions allows us to derive closed expressions for expectation values and the single particle Green's function of the minority particle. The outline of the chapter is as follows:

In Sec. 3.1, we collect the ingredients needed from the forgoing chapter and analyze the Bethe-Ansatz equations for the present case. Section 3.2 is devoted to the derivation of expectation values. The equal time single particle Green's function of the minority Fermion is calculated in Sec. 3.3. In Sec. 3.4, we study the equal time density-density correlation function. The survival probability and the local density of states are discussed in Sec. 3.5.

## 3.1 Eigenfunctions and Bethe-Ansatz equations

For one spin-up Fermion the Hamiltonian (2.2) of the system reads

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - \frac{\partial^2}{\partial y^2} + 4c \sum_{j=1}^{N} \delta(x_j - y) \ . \tag{3.1}$$

The corresponding eigenfunctions are determined by Theorem 1. With $M = 1$ we obtain

$$\begin{aligned} \Psi(\mathbf{x}, \mathbf{k}, y, \Lambda) &= C_N \det \left[ A_j(\Lambda, x_l - y) e^{\imath k_j x_l} \middle| e^{\imath k_j y} \right]_{\substack{j=1,\ldots,N+1 \\ l=1,\ldots,N}} \ , \\ A_j(\Lambda, x) &= \imath(k_j - \Lambda) + c\,\mathrm{sgn}(x) \ . \end{aligned} \tag{3.2}$$

Most conveniently the wave functions are in Eq. (3.2) expressed by a single determinant. Since there is no need to distinguish between different spin-up particles, we drop the indices of the corresponding quantities as done in Eqs. (3.1) and (3.2). Moreover, in the present context the statistics of the spin-up particle is irrelevant. Thats why we will often refer to it also as extra particle or distinguishable particle. We exclusively study the

case of periodic boundary conditions. In Sec. 3.3, we derive the short distance behavior of the extra particle's Green's function. In the course of doing so, the normalization constant $C_N$ for periodic boundary conditions is obtained, see Eq. (3.79). It reads

$$|C_N|^{-2} = N! L^{N+1} \prod_{j=1}^{N+1} \left[ (k_j - \Lambda)^2 + c^2 + \frac{2c}{L} \right] \sum_{l=1}^{N+1} \frac{1}{(k_l - \Lambda)^2 + c^2 + 2c/L} \; . \tag{3.3}$$

The quantization rules are given by the Bethe-Ansatz equations (2.32) and (2.33). For $M = 1$ they acquire the form

$$k_j L = 2\pi \left( m_j + \frac{1}{2} \right) - 2 \arctan \left( \frac{k_j - \Lambda}{c} \right) \; , \quad j = 1, \ldots, N+1 \; , \tag{3.4}$$

$$\pi J = \sum_{j=1}^{N+1} \arctan \left( \frac{k_j - \Lambda}{c} \right) \; , \tag{3.5}$$

where the quantum numbers $m_j$ and $J$ are integers and, according to Eq. (2.34), the spectrum of $J$ is bounded by the condition

$$-\frac{N+1}{2} \leq J \leq \frac{N+1}{2} \; . \tag{3.6}$$

In the hardcore limit, where $c \to \infty$, Eqs. (3.4) and (3.5) become

$$k_j = \frac{2\pi}{L} \left( m_j + \frac{1}{2} - \frac{J}{N+1} \right) \; , \tag{3.7}$$

$$\lambda = -\tan \left( \frac{\pi J}{N+1} \right) \; , \quad \lambda = \frac{\Lambda}{c} \; . \tag{3.8}$$

From the definition in Eq. (3.8) it is seen that $\lambda$ ranges from $-\infty$ to $+\infty$ as the quantum number $J$ varies according to Eq. (3.6).

For the ground state the quantum numbers $m_j$ are determined by Eq. (2.36)

$$\{m_j\}_{j=1,\ldots,N+1} = \left\{ \frac{N-1}{2}, \ldots, 0, \ldots, -\frac{N+1}{2} \right\} \; . \tag{3.9}$$

Together with $J = 0$ this yields in the limit of vanishing interaction strength the ground state solution

$$\Lambda = 0 \quad , \quad \{k_j\}_{j=1,\ldots,N+1} = \frac{2\pi}{L} \left\{ 0^+, 0^-, \pm 1, \ldots, \pm \frac{N-1}{2} \right\} \tag{3.10}$$

of the Bethe-Ansatz equations. In the following we assume that the spin-down particles are in the ground state such that their non-interacting momentum distribution is given by a Fermi-sea. Upon this we study excitations where the extra particle may occupy an arbitrary single particle state. Two cases have to be distinguished:

1. Either the quasi-momentum of the extra particle lies inside the Fermi-sea. Then the quantum numbers $m_j$ of the state are given by the set of integers in Eq. (3.9). The non-interacting extra particle shares its momentum with one of the particles of the Fermi-sea. The additional quantum number $J$ indicates the double occupied quasi-momentum. For $J = (N-1)/2$ or $J = -(N-1)/2$ the Fermi-momentum of the sea is double occupied. The non-interacting eigenfunctions for a given set of quasi-momenta $\{k_j\}_{j=1,\ldots,N+1}$ is uniquely determined and the energy-eigenvalue is non-degenerated. Thus for $c = 0$ the energy increases quadratically with $J$.

2. In the second case, the quasi-momentum of the extra particle lies outside the Fermi-sea. The quantum number $J$ takes its value at the upper edge $J = (N+1)/2$ or at the lower edge $J = -(N+1)/2$ of its spectrum. For vanishing interaction all $N+1$ quasi-momenta are different. If the highest/lowest quantum number is $m_1 = (N-1)/2$, respectively $m_{N+1} = -(N+1)/2$, the extra particle's momentum borders the Fermi-sea. If $m_1 > (N-1)/2$ or $m_{N+1} < -(N+1)/2$ the extra particle's momentum is outside the Fermi-sea. For a given set of quasi-momenta there exist $N+1$ orthogonal eigenfunctions, since the extra particle might carry any of the $k_n$'s without changing the system's energy. These eigenfunctions are distinguished by $J$.

The different cases are illustrated in Fig. 3.1. We always assume a dense set of quantum numbers as in Eq. (3.9). Hence the extra particle's momentum borders the Fermi-sea for $J = \pm(N+1)/2$ and lies inside if $|J| < (N+1)/2$. This corresponds to the first three cases shown in Fig. 3.1.

In the thermodynamic limit we assume the quasi-momenta to be distributed in between the two bounds $k_\pm$ as defined in Eq. (2.51). From Eq. (2.49) we find for $M = 1$ the corresponding density of states

$$\varrho(k) = \frac{L}{2\pi} + \frac{1}{\pi}\frac{c}{(k-\Lambda)^2 + c^2} \ . \tag{3.11}$$

The system's overall momentum $K$ is determined by Eq. (2.45). For the set of quantum numbers (3.9) and $M = 1$ we find that $K = -2\pi J/L$ and hence the quantum number $J$ is directly related to the system's overall momentum. The latter in turn is identified with the momentum of the extra particle in the lab frame where the Fermi-sea is at rest. In the thermodynamic limit it then proves useful to define the quantity $\hat{K} = K/k_{\mathrm{F}} = -2J/N$. It corresponds to the overall momentum in units of $k_{\mathrm{F}}$ and measures the location of the extra particle's momentum within the Fermi-sea. For $\hat{K} = 0$ the extra particle's momentum is in the center, for $\hat{K} = \pm 1$ it is at the lower or upper Fermi-edge.

As the eigenfunctions (3.2) are determinants, we often make use of the basic prop-

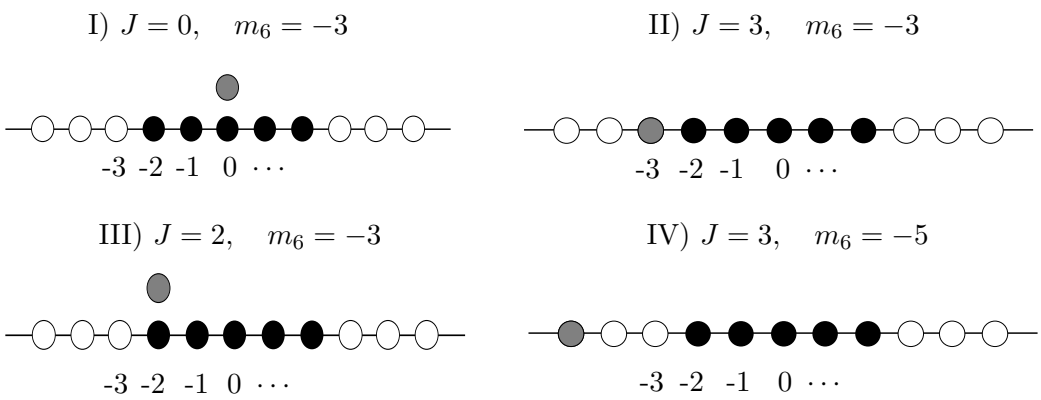


Figure 3.1: Sketch of the occupied non-interacting single particle states for $N = 5$. A black circle denotes a state occupied by a sea-Fermion, an empty circle denotes a non-occupied state. The gray circle represents the state, which is occupied by the extra particle. For the cases I)-III), the quantum numbers $m_j$ are given by Eq. (3.9). Case I) corresponds to the ground state configuration. In the second case the extra particle's momentum boarders the Fermi-sea. For case III) it is located right at the Fermi-edge. The case IV) illustrates the situation where the lowest quantum number $m_6 = -5$ and $J = 3$ such that the momentum of the extra particle is outside the Fermi-sea .

erties for determinants. In particular we will use the identity

$$\int dx_1 \ldots \int dx_N \det\left[f_j(x_l)\right]_{j,l=1,\ldots,N} \det\left[h_j(x_l)\right]_{j,l=1,\ldots,N} = N! \det\left[\int dx h_j(x) f_l(x)\right]_{j,l=1,\ldots,N} . \tag{3.12}$$

Here $f_j(x)$ and $h_j(x)$, $j = 1, \ldots, N$ are arbitrary functions. The relation above can easily be proven by using the permutation invariance of the integration variables and basic properties of the symmetric group.

## 3.2 Expectation values

In Sec. 3.2.1, we study the energy shift and the extra particle's energy in presence of the Fermi-sea. The mean interaction energy and the expectation value for the kinetic energy of the extra particle are calculated in Secs. 3.2.2 and 3.2.3, respectively.

### 3.2.1 Energy-shift

Using the density of states (3.11), the system's total energy can be written as

$$E = \int_{-k_-}^{+k_+} dk \varrho(k) k^2 = \frac{L(k_+^3 - k_-^3)}{6\pi} + \frac{c}{\pi} \int_{-k_-}^{+k_+} dk \frac{k^2}{(k-\Lambda)^2 + c^2} , \tag{3.13}$$

where the boundaries $k_\pm$ are determined by Eq. (2.51). Our aim is to evaluate this expression up to terms of order $L^{-1}$. To do so, we consider the both contributions on the right hand side separately.

The first term in Eq. (3.13) scales like $L$. Therefore corrections to $k_\pm^3$ up to the order $L^{-1}$ must be taken into account. Using the definition (2.51) for $k_\pm$ we obtain

$$\frac{L(k_+^3 - k_-^3)}{6\pi} = E_\mathrm{F} + k_\mathrm{F}^2 - \frac{k_\mathrm{F}^2}{\pi} v(\Lambda, k_\mathrm{F}) + \mathcal{O}(L^{-1}) \ , \tag{3.14}$$

where $E_\mathrm{F} = Lk_\mathrm{F}^3/(3\pi)$ denotes the energy of the non-interacting Fermi-sea and $k_\mathrm{F} = \pi N/L$ is the Fermi-momentum. Since it appears frequently throughout the following, we introduced in Eq. (3.14) the notation

$$v(\Lambda, k_\mathrm{F}) = \left[ \arctan\left(\frac{k_\mathrm{F} - \Lambda}{c}\right) + \arctan\left(\frac{k_\mathrm{F} + \Lambda}{c}\right) \right] \ . \tag{3.15}$$

The second term in Eq. (3.13) is of order one and hence it is sufficient to consider the upper and lower bounds of the momentum distribution in leading order i.e. $k_\pm = \pm k_\mathrm{F}$. The corresponding integral can be evaluated.

On the other hand, the energy of the non-interacting system reads

$$E_0 \ = \ E_\mathrm{F} + K^2 \ . \tag{3.16}$$

To obtain the energy shift due to the interaction, we take the difference of Eqs (3.13) and (3.16). Doing so, the contribution due to the Fermi-sea drops out and we obtain

$$E - E_0 \ = \ E^{(1)}(\Lambda) - K^2 \ , \tag{3.17}$$

where

$$E^{(1)}(\Lambda) = k_\mathrm{F}^2 - \frac{k_\mathrm{F}^2 - \Lambda^2 + c^2}{\pi} v(\Lambda, k_\mathrm{F}) + \frac{c}{\pi} \left[ 2k_\mathrm{F} + \Lambda \ln\left( \frac{c^2 + (\Lambda - k_\mathrm{F})^2}{c^2 + (\Lambda + k_\mathrm{F})^2} \right) \right] \ . \tag{3.18}$$

This term has a natural interpretation as the extra particle's energy in the presence of the Fermi-sea. The energy shift is obtained by subtracting the extra particle's non-interacting energy i.e. $K^2$.

In Eq. (3.18), the function $E^{(1)}(\Lambda)$ is expressed in terms of $c, k_\mathrm{F}$ and $\Lambda$. However, the parameter $\Lambda$ is not independent but rather determined by $c, k_\mathrm{F}$ and $K$. The corresponding relation is provided by the second Bethe-Ansatz equation. In the thermodynamic limit it reads (compare Eq. (2.53))

$$K \ = \ -\frac{1}{\pi} \int_{-k_\mathrm{F}}^{+k_\mathrm{F}} dk \arctan\left(\frac{k - \Lambda}{c}\right) + \mathcal{O}(1/L) \ . \tag{3.19}$$

From this equation follows in particular that $\Lambda|_{c=0} = K$. Although the integral in Eq. (3.19) can be evaluated, the resulting transcendental equation can not be solved analytically for $\Lambda$. Since the wave function explicitly depends on $\Lambda$, the issue of expressing

$\Lambda$ in terms of the three independent $c, k_\mathrm{F}$ and $K$ is not restricted to the present case but will be encountered for any quantity derived from the exact eigenfunctions. However, numerically this problem can easily implemented and Eq. (3.19) be inverted.

For the ground state the situation is more favorable, since the solution of Eq. (3.19) can be made explicit. In this case we have $K = \Lambda = 0$ and the energy shift coincides with the extra particle's energy $E^{(1)}(\Lambda)$. From Eqs. (3.17) and (3.18) we obtain

$$E^{(1)}(0) = E - E_0 = \frac{2}{\pi}\left[ck_\mathrm{F} + k_\mathrm{F}^2 \arctan\left(\frac{c}{k_\mathrm{F}}\right) - c^2 \arctan\left(\frac{k_\mathrm{F}}{c}\right)\right]. \tag{3.20}$$

In Fig. 3.2, we show the plot of the energy shift versus the interaction strength for different values of $\hat{K}$. As expected, the energy shift tends to zero for vanishing interaction strength. Expanding Eq. (3.17) in the strong coupling limit reveals that the energy shift saturates at a value $E_\mathrm{max} = k_\mathrm{F}^2 - K^2$ for $c \to \infty$. In between these two values the curve increases monotonically.

A plot of the energy $E^{(1)}(\Lambda)$ as function of the extra particle's free momentum $K$ is shown Fig. 3.3. It is seen that, at least for small values of $c$ and $K$, the energy still is quadratic. It thus might be approximated by

$$E_1(k) \approx \Sigma + \frac{k^2}{2m^*}\ , \tag{3.21}$$

where $m^*$ and $\Sigma$ are interpreted as the effective mass and the self-energy of the extra particle in the presence of the Fermi-sea, respectively. Expanding the expression (3.18) up to the quadratic term in $K$, they can be identified as

$$\Sigma = \frac{1}{\pi}\left(2ck_\mathrm{F} + 2(k_\mathrm{F}^2 + c^2)\arctan\left(\frac{c}{k_\mathrm{F}}\right) - \pi c^2\right)\ , \tag{3.22}$$

$$m^* = \frac{1}{2}\frac{\pi/2 - 2\arctan(c/k_\mathrm{F}) + (2/\pi)\arctan^2(c/k_\mathrm{F})}{\pi/2 - \arctan(c/k_\mathrm{F}) - c(k_\mathrm{F}^2 + c^2)/k_\mathrm{F}^3}\ . \tag{3.23}$$

For finite interaction strength $c > 0$ and for larger momenta $K$ the dispersion relation deviates from the quadratic behavior. In the strong coupling limit, the self-energy tends to $k_\mathrm{F}^2$ and the mass becomes formally infinite, leading to a momentum independent dispersion relation. In particular for $K = k_\mathrm{F}$ the first derivative vanishes, yielding a van Hove like singularity.

### 3.2.2 Interaction energy

We consider the mean interaction energy

$$\langle\hat{V}\rangle = \langle\Psi|4c\sum_{n=1}^{N}\delta(x_n - y)|\Psi\rangle\ . \tag{3.24}$$

To calculate it we use Pauli's trick (see Ref. [72]), that is

$$\langle\hat{V}\rangle = c\frac{dE}{dc}\ . \tag{3.25}$$

In the ground state the right hand side of Eq. (3.25) can be evaluated by using the expression (3.20). This yields for the mean interaction energy of the ground state

$$\langle \hat{V} \rangle \;=\; \frac{4c}{\pi} \left[ k_{\mathrm{F}} - c \arctan\left( \frac{k_{\mathrm{F}}}{c} \right) \right] \;. \tag{3.26}$$

The equation above reveals that $\langle \hat{V} \rangle$ vanishes like $c^{-1}$ for $c \to \infty$. Consequently the interaction energy vanishes in the weak coupling limit as well as in the strong coupling limit.

For exited states with $K \neq 0$, the derivative in Eq. (3.25) can not be carried out directly due to the implicit dependence of $\Lambda$ on $c$ via Eq. (3.19). This problem might be outmaneuvered by the following observation: Since the interaction potential is local, the interaction energy is given by the interaction strength times the probability to find a sea-particle at the same position as the extra particle. The latter quantity is identical with the equal time density-density correlation function $R(x, y)$ at $x = y = 0$. Hence, by definition, the interaction energy relates to the local density-density correlation function $R(0, 0)$ via

$$\langle \hat{V} \rangle \;=\; 4c\, L\, R(0,0) \;. \tag{3.27}$$

Thus given $\langle \hat{V} \rangle$, we can calculate $R(0, 0)$ and vice versa. For example, combining Eq. (3.26) with (3.27) yields the ground state expression for $R(0, 0)$

$$R(0,0) \;=\; \frac{N\pi}{k_{\mathrm{F}}} \left[ 1 - \frac{c}{k_{\mathrm{F}}} \arctan\left( \frac{k_{\mathrm{F}}}{c} \right) \right] . \tag{3.28}$$

The other way around: knowing a general expression for $R(0, 0)$ we obtain via Eq. (3.27) the interaction energy for exited states with $K \neq 0$. In Sec. 3.4, where the full density–density correlation function is studied, we derive an expression of $R(0, 0)$ for $K \neq 0$, see Eq. (3.95). This allows us to calculate the interaction energy in this case as well. Equating Eqs. (3.27) and (3.95) we obtain

$$\langle \hat{V} \rangle \;=\; \frac{4c}{\pi} \left( k_{\mathrm{F}} - \frac{c}{2v(\Lambda, k_{\mathrm{F}})} \left| \ln\left( \frac{c + \imath(k_{\mathrm{F}} - \Lambda)}{c - \imath(k_{\mathrm{F}} + \Lambda)} \right) \right|^2 \right) \;. \tag{3.29}$$

Note that for the ground state, where $K = \Lambda = 0$, this simplifies to Eq. (3.26).

In Fig. 3.2, the energy shift and the interaction energy are plotted versus the interaction strength for three different values of $\hat{K}$. It is seen that the interaction energy has a maximum as predicted. On the other hand the energy shift increases monotonously and saturates at a value $E_{\max}/k_{\mathrm{F}}^2 = 1 - \hat{K}^2$. As $K$ approaches the Fermi-edge, the energy shift as well as the mean interaction energy become smaller and smaller and finally, for the extra particle's momentum just at $k_{\mathrm{F}}$, they vanish identically for arbitrary interaction strength. In that sense, a extra particle with a momentum at the Fermi-edge behaves like an additional particle of the Fermi-sea.

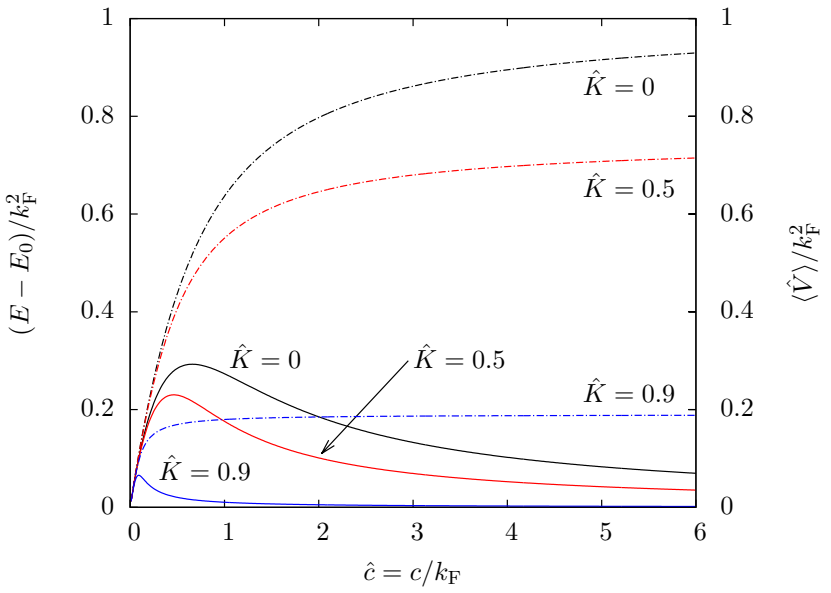


Figure 3.2: Interaction energy (full lines) and energy shift (dot-dashed lines) as function of $\hat{c}$. The values of $\hat{K}$ are $\hat{K}=0$ (black), $\hat{K}=0.5$ (red) and $\hat{K}=0.9$ (blue).

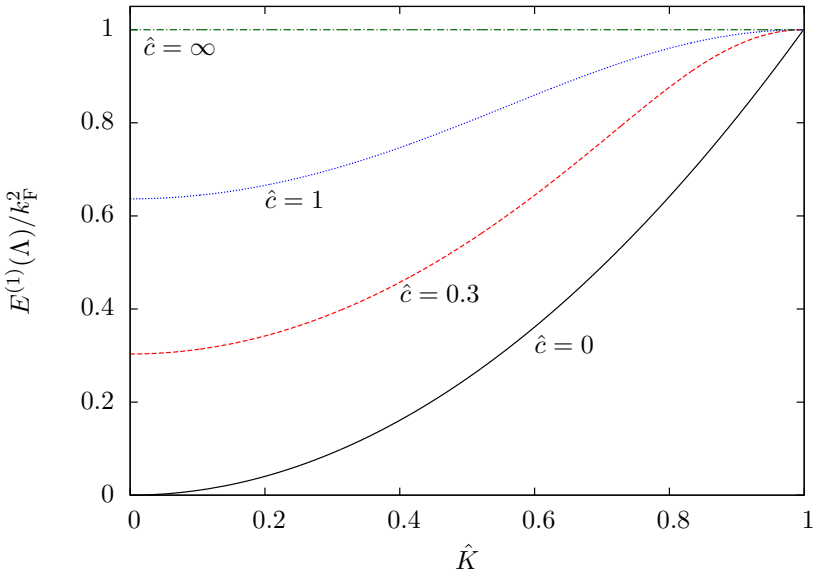


Figure 3.3: Single particle energy of the extra particle as function of its free momentum $\hat{K}=K/k_F$. The values of $\hat{c}=c/k_F$ are $\hat{c}=0$ (solid line, black), $\hat{c}=0.3$ (dashed line, red), $\hat{c}=1$ (dotted line, blue) and $\hat{c}=\infty$ (dot-dashed line, green).

### 3.2.3 Kinetic energy of the extra particle

Using the wave function (3.2), the expectation value of the extra particle's kinetic energy can be calculated. Our starting point is the corresponding expression in coordinate representation

$$\begin{aligned}\langle \hat{p}_y^2 \rangle &= \int_0^L dy \int_0^L dx_1 \cdots \int_0^L dx_N \Psi^*(\mathbf{x},\mathbf{k},y,\Lambda) \left(-\frac{\partial^2}{\partial y^2}\right) \Psi(\mathbf{x},\mathbf{k},y,\Lambda) \qquad (3.30)\\ &= \int_0^L dy \int_0^L dx_1 \cdots \int_0^L dx_N \left|\frac{\partial}{\partial y}\Psi(\mathbf{x},\mathbf{k},y,\Lambda)\right|^2 ,\end{aligned}$$

where the second line is obtained via integration by parts with respect to $y$. From Eq. (3.2) we find for the first derivative of the wave function with respect to $y$

$$\frac{\partial}{\partial y}\Psi(\mathbf{x},\mathbf{k},y,\Lambda) = C_N \prod_{n=1}^{N+1} e^{\imath k_n y} \det\left[A_j(\Lambda, x_l - y) e^{\imath k_j (x_l - y)} \Big| \imath k_j\right]_{\substack{j=1,\ldots,N+1 \\ l=1,\ldots,N}} . \qquad (3.31)$$

Thus the integrand in Eq. (3.30) is a function of the difference of variables $x_l - y$ only. We employ translational invariance and shift the variables $x_l \to x_l + y$. In order to carry out the integration we us the identity (3.12) as well as

$$\begin{aligned}\int_0^L dx A_j(x) A_l^*(x) e^{\imath(k_j - k_l)x} &= A_j(1) A_l^*(1) \frac{e^{\imath(k_j-k_l)L} - 1}{\imath(k_j - k_l)} \qquad (3.32)\\ &= L\Big((k_j - \Lambda)^2 + c^2\Big)\delta_{jl} + \mathcal{O}(N^{-1}) ,\end{aligned}$$

where the Bethe-Ansatz equation (3.4) has been used. After some further algebra we find

$$\langle \hat{p}_y^2 \rangle = |C_N|^2 L^{N+1} N! \prod_{j=1}^{N+1} |A_j(\Lambda, 1)|^2 \sum_{n=1}^{N+1} \frac{k_n^2}{(k_n - \Lambda)^2 + c^2} . \qquad (3.33)$$

Taking into account the normalization constant (3.3), the expression simplifies further. In the thermodynamic limit the summations in Eq. (3.33) might be replaced by integrals using the density of states (3.11). The resulting integrations can be done and in leading order of the system size we obtain

$$\langle \hat{p}_y^2 \rangle = \Lambda^2 - c^2 + v^{-1}(\Lambda, k_\mathrm{F}) c \left(2k_\mathrm{F} + \Lambda \ln\left(\frac{c^2 + (k_\mathrm{F} - \Lambda)^2}{c^2 + (k_\mathrm{F} + \Lambda)^2}\right)\right) + \mathcal{O}(L^{-1}) . \qquad (3.34)$$

As in the former cases, the implicit dependence of $\Lambda$ on $c$ prevents to express Eq. (3.34) in terms of the three independent quantities $c, k_\mathrm{F}$ and $K$. However, numerically it can

easily be implemented. In particular for the ground state, where $\Lambda \equiv 0$, the expression above simplifies to

$$\langle \hat{p}_y^2 \rangle = c \left( \frac{k_\mathrm{F}}{\arctan(k_\mathrm{F}/c)} - c \right). \tag{3.35}$$

We discuss the limiting behavior of $\langle \hat{p}_y^2 \rangle$. As is readily seen Eq. (3.34) yields for vanishing interaction strength $\langle \hat{p}_y^2 \rangle = \Lambda^2 = K^2$. This is expected since $\Lambda\big|_{c=0}$, respectively $K$ correspond to free momentum of the extra particle. On the other hand, the kinetic energy of the extra particle approaches for infinite strong interaction strength the asymptotic value $k_\mathrm{F}^2/3 = E_\mathrm{F}/N$, which is identified as the energy per particle of the Fermi-sea. In Fig. 3.4, we show the plot of $\langle \hat{p}_y^2 \rangle / k_\mathrm{F}^2$ as function of the interaction strength. Depending on its initial momentum for $c = 0$, the extra particle gains or losses kinetic energy through the interaction with the Fermi-sea. Moreover it is seen that the closer $K$ comes to the Fermi-edge, the faster the asymptotic value is reached.

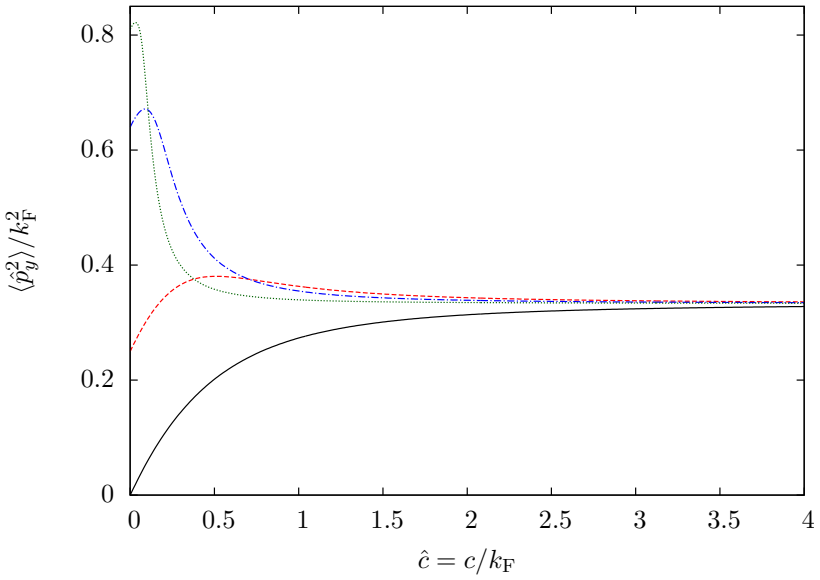


Figure 3.4: Kinetic energy of the extra particle versus the interaction strength for different values of its initial momentum $\hat{K} = K/k_\mathrm{F}$. The values of $\hat{K}$ are $\hat{K} = 0$ (solid line, black), $\hat{K} = 0.5$ (dashed line, red), $\hat{K} = 0.8$ (dot-dashed line, blue) and $\hat{K} = 0.9$ (dotted line, green).

## 3.3 Equal time Green's function

We consider the equal time Green's function of the extra particle. Based on the reformulation of the exact eigenfunctions, we derive in Sec. 3.3.1 a determinantal expression

for the Green's function for a finite number of particles. In Sec. 3.3.2, the hardcore limit is analyzed. The case of finite interaction strength is studied in Sec. 3.3.3.

### 3.3.1 Determinantal representation

In coordinate representation the equal time Green's function of the extra particle is defined as the $N$-fold integral

$$G(y,y') = \int_0^L dx_1 \cdots \int_0^L dx_N \Psi(\mathbf{x},\mathbf{k},y,\Lambda)\Psi^*(\mathbf{x},\mathbf{k},y',\Lambda) \ . \tag{3.36}$$

Likewise it can be interpreted as the reduced density matrix of the extra particle. We stick to the notation *equal time Green's function* or simply *Green's function*. The normalization of the wave function to unity implies

$$\int_0^L dy G(y,y) = 1 \ . \tag{3.37}$$

Furthermore we deduce from Eq. (3.36) that exchanging the arguments of $G(y,y')$ and taking the complex conjugated yields the identity i.e.

$$G(y,y') = \left[G(y',y)\right]^* \ . \tag{3.38}$$

This implies that the real part of $G(y,y')$ is symmetric, whereas the imaginary part is antisymmetric. Our starting point for the evaluation of $G(y,y')$ is the determinantal form of the exact wave functions. Using them closed expressions for $G(y,y')$ can be obtained.

We expand the determinant in Eq. (3.2) with respect to the last column and substitute the resulting expression into Eq. (3.36). This yields

$$\begin{aligned} G(y,y') &= |C_N|^2 \sum_{n=1}^{N+1}\sum_{m=1}^{N+1}(-1)^{n+m} e^{\imath(k_n y - k_m y')} \\ &\quad \times \int_0^L dx_1 \ldots \int_0^L dx_N \det\left[A_j(\Lambda, x_l - y)e^{\imath k_j x_l}\right]_{\substack{j=1,\ldots,N+1\neq n \\ l=1,\ldots,N}} \\ &\quad \times \det\left[A_j^*(\Lambda, x_l - y')e^{-\imath k_j x_l}\right]_{\substack{j=1,\ldots,N+1\neq m \\ l=1,\ldots,N}} . \end{aligned} \tag{3.39}$$

Now the particular form of the eigenfunctions allows us to employ the identity (3.12). Doing so, one determinant in Eq. (3.39) can be replaced by its diagonal part and the $x$-integrations can be performed. The resulting expression acquires the form

$$G(y,y') = |C_N|^2 \, N! \sum_{n=1}^{N+1}\sum_{m=1}^{N+1}(-1)^{n+m} e^{\imath(k_n+k_m)(y-y')/2} \det\left[K_{jl}(y,y')\right]_{\substack{j=1,\ldots,N+1\neq n \\ l=1,\ldots,N+1\neq m}} , \tag{3.40}$$

where the matrix entries are given by

$$K_{jl}(y,y') = e^{-\imath(k_j-k_l)(y+y')/2} \int_0^L dx A_j(\Lambda, x-y) A_l^*(\Lambda, x-y') e^{\imath(k_j-k_l)x} \ . \tag{3.41}$$

The evaluation of $K_{jl}(y^-)$ is straightforward but tedious. The corresponding calculation can be found in Sec. A.3 of Appendix A. It yields

$$\begin{aligned} K_{jl}(y^-) &= \left[L((k_j-\Lambda)^2+c^2) + 2c\imath(k_j-\Lambda)y^- - 2c^2|y|^-)\right]\delta_{jl} \\ &-2c\left\{\cos\left(\frac{(k_j-k_l)y^-}{2}\right) - \imath\frac{k_l+k_j-2\Lambda+\imath 2c\,\mathrm{sgn}(y^-)}{k_j-k_l}\sin\left(\frac{(k_j-k_l)y^-}{2}\right)\right\}(1-\delta_{ij}) \end{aligned} \tag{3.42}$$

with $y^- = (y-y')$. Equation (3.42) makes explicit that $G(y,y')$ is a function of the difference $y^-$ only, as expected by translation invariance. The expression (3.40) can be cast into the more compact form

$$G(y,y') = e^{\imath K y^-}\,|C_N|^2\,N!\left[\begin{array}{c|c} [1]_{n=1,\ldots,N+1} & 0 \\ \hline [g_{nm}]_{n,m=1,\ldots,N+1} & [1]_{m=1,\ldots,N+1} \end{array}\right] . \tag{3.43}$$

Here we have introduced the notation

$$g_{nm} = e^{-\imath(k_n+k_m)y^-/2} K_{nm}(y^-) \ . \tag{3.44}$$

By appropriately adding the rows and columns of the determinant in Eq. (3.43), the right hand side can be expressed in terms of a single $N\times N$ determinant

$$G(y,y') \;=\; \frac{e^{\imath K y^-}}{L} G_{\mathrm{I}}(y^-) \ , \tag{3.45}$$

where we have introduced the interaction part $G_{\mathrm{I}}(y^-)$ of the Green's function

$$G_{\mathrm{I}}(y^-) = |C_N|^2\,N!L\det\left[g_{nm} - g_{nm+1} - g_{n+1m} + g_{n+1m+1}\right]_{n,m=1,\ldots,N} \ . \tag{3.46}$$

According to the normalization condition (3.37), we have $G_{\mathrm{I}}(0) = 1$. Separating the overall momentum and $G_{\mathrm{I}}(y^-)$ as in Eq. (3.45) is useful in order to distinguish the oscillations due to the exponential in Eq. (3.45) from those due to the interaction. The representation of the Green's function as a determinant is most convenient for a further numerical analysis.

In Figs. 3.5 and 3.6 the real part of $G_{\mathrm{I}}(y^-)$ is plotted for a particle number of $N = 15$. The quantum numbers are chosen according to Eq. (3.9) and the interaction strength has the scaling $\bar{c} = cL$. Figure 3.5 shows the transition of the interaction part of the Green's function for the ground state as $\bar{c}$ varies from $\bar{c} = 0$ to $\bar{c} = \infty$. In Fig. 3.6, we show $G_{\mathrm{I}}(y^-)$ for exited states with $J \neq 0$ and fixed interaction strength $\bar{c} = 50$. Whereas in

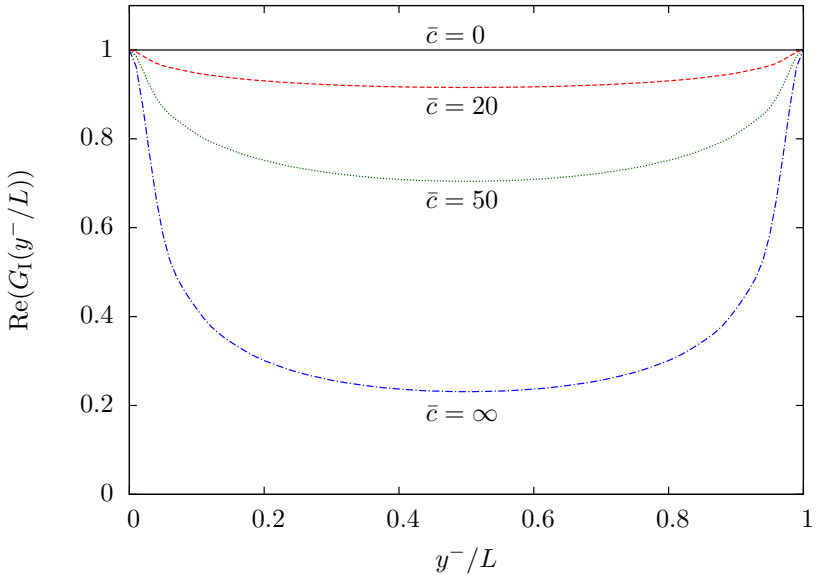


Figure 3.5: Real part of $G_\mathrm{I}(y^-)$ as function $y^-/L$ for the ground state i.e. $J = 0$ and $N = 15$. The values of $\bar{c}$ are $\bar{c} = 0$ (full line, black), $\bar{c} = 20$ (dashed line, red), $\bar{c} = 50$ (dotted line, green) and $\bar{c} = \infty$ (dot-dashed line, blue).

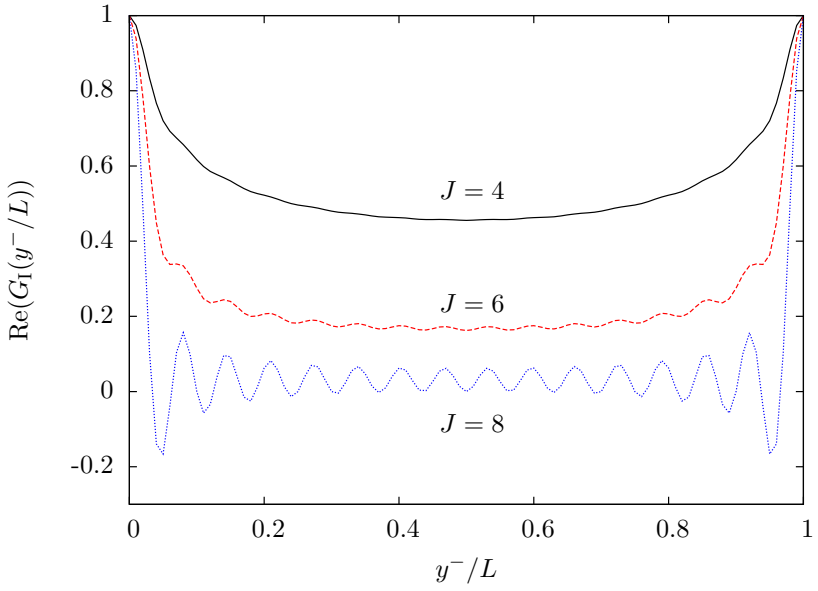


Figure 3.6: Real part of $G_\mathrm{I}(y^-)$ as function $y^-/L$ for $\bar{c} = 50$ and $N = 15$. The values of $J$ are $J = 4$ (solid line, black), $J = 6$ (dashed line, red), and $J = 8$ (dotted line, blue).

the ground state the curve is smooth and decays monotonously up to $y^-/L = 0.5$, for the excited states wiggles develop and the function seems to become oscillatory. However, only for the highest value $J = 8$, i.e. if the extra particle's momentum just borders the Fermi-sea, the curve has nodes.

In the following, we analyze the representation (3.46) of $G_{\rm I}(y^-)$ further in the hard-core limit. We show that the features, shown in Figs. 3.5 and 3.6 and described above persist in the thermodynamic limit.

### 3.3.2 Hardcore limit

We turn to the limit of infinite strong interaction strength. Our starting point is the wave function (3.2). For $c \to \infty$ it acquires the form

$$\Psi(\mathbf{x}, \mathbf{k}, y, \lambda) \propto \prod_{n=1}^{N} \Big( -\imath\lambda + \mathrm{sgn}(x_n - y) \Big) \det \Big[ e^{\imath k_j x_l} \Big| e^{\imath k_j y} \Big]_{\substack{j=1,\ldots,N+1 \\ l=1,\ldots,N}} \quad , \tag{3.47}$$

where we recall that $\lambda = \tan(\pi \hat{K}/2)$ directly relates to the momentum $\hat{K} = K/k_{\rm F}$ of the extra particle. The case $\lambda = 0$ corresponds to $\hat{K} = 0$, while for $\lambda = \pm\infty$ the extra particle's momentum is at the edges of the Fermi-sea i.e. $\hat{K} = \pm 1$

For $\lambda = 0$ this expression resembles the wave function of $N + 1$ hardcore Bosons [24]. The only difference consists in the product of sign-functions: The wave function for hardcore Bosons is a Slater determinant multiplied not with a single product of sign-functions but with a double product $\prod_{i<j} \mathrm{sgn}(x_i - x_j)$. For the equal time Green's function this difference is irrelevant and it is identical with those of hardcore Bosons. A large body of literature has been devoted to the studies of the latter [56, 73, 74, 75, 55, 53, 54, 76, 77, 78].

On the other hand for $|\lambda| \to \infty$, the wave function and therefore also the Green's function coincides with that of free Fermions. This is already reflected in the Bethe-Ansatz equations (3.4) and (3.5). The quasi-momenta $k_n$ are multiple integers of $2\pi/L$ with an offset $\arctan(\lambda)/\pi - 1/2$, that varies from $-1/2$ for $\lambda = 0$ to 0 for $\lambda = \pm\infty$.

Hence, varying $\lambda$ from $\lambda = 0$ to $\lambda = \infty$, we expect the extra particle's Green's function to undergo a transition from the one of impenetrable Bosons to that of free Fermions. As there are two fundamental formulations of the Green's function for impenetrable Bosons we will go for two different ways to study this transition.

The first method is based on the theory of Töplitz determinants [56, 73, 79, 53] and we will review it in Sec. A.4 of Appendix A. The starting point for this approach is the representation (3.46). Taking the limit $c \to \infty$ of Eq. (3.46) yields a determinantal form of the Green's function which depends on $\lambda$. While for $\lambda = 0$ the corresponding expression coincides with the Töplitz determinant found for the Green's function of impenetrable Bosons, it yields for $\lambda = \pm\infty$ the Green's function of free Fermions.

Here we discuss an alternative approach. It is based on the description of the Green's function via solutions of a Painlevé equation [55, 54]. We show that the above mentioned transition is for zero temperature described by a change in the initial condition of the solution of one and the same Painlevé equation.

We substitute Eq. (3.47) into the definition (3.36) of the equal time Green's function. This yields

$$G(y,y') = |C_N|^2 \int_0^L dx_1 \cdots \int_0^L dx_N \prod_{l=1}^{N} (\imath\lambda + \mathrm{sgn}(x_l - y))(-\imath\lambda + \mathrm{sgn}(x_l - y')) \quad (3.48)$$
$$\times \det\left[e^{\imath k_j x_l}\middle| e^{\imath k_j y}\right]_{\substack{j=1,\ldots,N+1\\ l=1,\ldots,N}} \det\left[e^{-\imath k_j x_l}\middle| e^{-\imath k_j y'}\right]_{\substack{j=1,\ldots,N+1\\ l=1,\ldots,N}} .$$

The crucial point is that for zero temperature, or – more precisely – if the Fermi-sea is in the ground state, the integral can be interpreted as an average over the unitary group. We rewrite the last line of the equation above according to

$$\det\left[e^{\imath k_j x_l}\middle| e^{\imath k_j y}\right]_{\substack{j=1,\ldots,N+1\\ l=1,\ldots,N}} \det\left[e^{-\imath k_j x_l}\middle| e^{-\imath k_j y'}\right]_{\substack{j=1,\ldots,N+1\\ l=1,\ldots,N}} \quad (3.49)$$
$$\propto \exp\left(\frac{2\imath}{L}\arctan(\lambda)y^-\right)\prod_{j<l}^{N}\sin^2\left(\frac{\pi(x_j-x_l)}{L}\right)\prod_{l=1}^{N}\sin\left(\frac{\pi x_l}{L}-\frac{\pi y}{L}\right)\sin\left(\frac{\pi x_l}{L}-\frac{\pi y'}{L}\right).$$

In the present case the first product can be identified with the measure of the unitary group $U(N)$ [80]

$$d\mu(U) \;\propto \prod_{j<l}^{N}\sin^2\left(\frac{\pi(x_j-x_l)}{L}\right)\prod_{j=1}^{N}\frac{2\pi dx_j}{L} \; . \quad (3.50)$$

The Green's function can again be written like in Eq. (3.45), where the part $G_{\mathrm{I}}$, which includes the effects of interaction, is now identified with the average of the function

$$\prod_{l=1}^{N}\left(1-\xi\chi^{(l)}_{[y',y)}\right)2\sin(\pi x_l)\left(-e^{-\frac{\pi\imath}{L}(x_l-y^-)}\right)\left(1-e^{\frac{2\pi\imath}{L}(x_l-y^-)}\right) \quad (3.51)$$

with respect to the measure $d\mu(U)$. Here we have introduced the indicator function

$$\chi^{(l)}_{[y',y)} = \begin{cases} 1 & \text{for} \quad x_l \in [y',y) \\ 0 & \text{else} \end{cases} \quad (3.52)$$

and the parameter

$$\xi = \frac{2}{1+\imath\lambda} \; . \quad (3.53)$$

Averages over the unitary group of functions of the type (3.51) were studied by Forrester, Frankel, Garoni and Witte [81, 53, 54] and they have been related to solutions of the Painlevé VI non-linear differential equation. Defining

$$u \;=\; \exp\left(-\imath\frac{2\pi y^-}{L}\right), \quad (3.54)$$

we follow Proposition 3 and Corollary 1 of Ref. [54] in order to deduce the relation

$$u(u-1)\frac{d}{du}\ln G_{\rm I}(y^-)\Big|_{u=e^{-\imath 2\pi y^-/L}} = \sigma_{N+1}(u) \tag{3.55}$$

between the interaction part of the Green's function and $\sigma_{N+1}(u)$. The function $\sigma_N(u)$ is a solution of the particular Painlevé VI equation due to Okamato [82]

$$\begin{aligned} -u^2(u-1)^2(\sigma_N'')^2 &= \big(\sigma_N-(u-1)\sigma_N'+1\big) \\ &\quad\times\left[4\sigma_N'(\sigma_N-u\,\sigma_N')-(N^2-1)(\sigma_N-(u-1)\sigma_N')\right]. \end{aligned} \tag{3.56}$$

The initial conditions are fixed by expanding $G_{\rm I}(y^-)$ for small distances like in Ref. [54]. We find that in terms of the variable $u$ the solution of Eq. (3.56), quested for, has the small distance expansion

$$\sigma_N(u) = \frac{N^2-1}{12}(u-1)^2 + \frac{\big(N^2-1\big)\left(\imath N/(1+\imath\lambda)-\pi\right)}{24\pi}(u-1)^3+\cdots\ . \tag{3.57}$$

Hence the initial conditions of $\sigma_N(u)$ explicitly depend on $\lambda$ and therefore on the free momentum of the extra particle.

One merit of formulating the Green's function in terms of Painlevé equations is that thermodynamic limit can easily be taken. First, Painlevé VI converts in the limit $N\to\infty$ to the Jimbo-Miwa-Okamoto form [55, 83] of the Painlevé V differential equation

$$\frac{x^2}{4}(\sigma_V'')^2 = \big(\sigma_V - x\sigma_V'+1\big)\left((\sigma_V')^2-\sigma_V+x\sigma_V'\right)\ , \tag{3.58}$$

where $x=k_{\rm F}y^-$. Second, the initial condition for $\sigma_V(x)$ becomes

$$\sigma_V(x) \sim -\frac{x^2}{3}+\frac{1}{1+\imath\lambda}\frac{x^3}{3\pi}+\cdots\ , \qquad x\to 0\ . \tag{3.59}$$

Thus the full Green's function reads in the thermodynamic limit

$$G(x) = \frac{e^{\imath\hat{K}x}}{L}G_{\rm I}(x) \qquad \text{with} \qquad G_{\rm I}(x)=\exp\left(\imath x\hat{K}+\int_0^x dx'\frac{\sigma_V(x')}{x'}\right)\ . \tag{3.60}$$

The solution of $\sigma_V$ and therefore also the Green's function, are uniquely determined by Eq. (3.58) and the initial conditions (3.59).

A second advantage of the formulation of the Green's function in terms of Painlevé equations is that there are established connection formulas which relate the short distance boundary conditions (3.59) to the long distance behavior of the solution. The case of Painlevé V has been solved by McCoy and Tang [84, 85, 86]. We employ the results obtained in Ref. [85] for hardcore Bosons:

Let the solution of Painlevé V as given in Eq. (3.58) be regular at $x=0$ with the expansion

$$\sigma_V(x) \sim -\frac{x^2}{3}+\frac{\xi}{6\pi}\,x^3+\dots\ , \qquad x\to 0\ . \tag{3.61}$$

Then for general $\xi \in \mathbb{C}$ the asymptotic expansion for $x \to \infty$ is different for $\xi \in (1, \infty)$ (case I) and for $\xi \in \mathbb{C} \setminus (1, \infty)$ (case II). In the first case it its given by

$$\begin{aligned} \sigma_V(x) &= 2k_{\rm I}x + 2k_{\rm I}^2 - \frac{1}{2} + \\ &\quad \frac{1}{4x}\left[\left(4k_{\rm I}^2 - 1\right)\sin 2s_{\rm I}(x) + 2k_{\rm I}\cos 2s_{\rm I}(x) - 2k_{\rm I}\left(4k_{\rm I}^2 + 1\right)\right] \\ &\quad -\frac{1}{x^2}\left[k_{\rm I}\left(4k_{\rm I}^2+1\right)^2 \sin 2s_{\rm I}(x) - M_{\rm I}\right] + \mathcal{O}(x^{-3}) \ , \end{aligned} \tag{3.62}$$

$$s_{\rm I}(x) = x + \phi_{\rm I} + 2k_{\rm I}\ln(x) \ , \tag{3.63}$$

where the parameters $k_{\rm I}, \phi_{\rm I}$ depend on $\xi$ as

$$\begin{aligned} k_{\rm I} &= \frac{1}{2\pi}\ln(\xi - 1) \ , \\ e^{2\imath\phi_{\rm I}} &= -2^{4\imath k_{\rm I}}\frac{\Gamma^2(-\imath k_{\rm I} + 1/2)}{\Gamma^2(\imath k_{\rm I} + 1/2)} \end{aligned} \tag{3.64}$$

and $M_{\rm I}$ is a constant to be determined. In case II, the asymptotic expansion is

$$\begin{aligned} \sigma_V(x) &= x\left(\cot s_{\rm II}(x) + 2k_{\rm II}\right) \\ &\quad +\frac{1}{2}\left(4k_{\rm II}^2 - 2\right) + \frac{3k_{\rm II}^2+1}{\left(\sin s_{\rm II}(x)\right)^2} + 2k_{\rm II}\cot s_{\rm II}(x) \\ &\quad +\frac{1}{x}\left[\frac{(1+3k_{\rm II}^2)^2\cot s_{\rm II}(x)}{(\sin s_{\rm II}(x))^2} + (1+3k_{\rm II}^2)\cot s_{\rm II}(x) + \right. \\ &\qquad \left. \frac{L_{\rm II}}{(\sin s_{\rm II}(x))^2} + M_{\rm II}\right] + \mathcal{O}(x^{-2}) \ , \end{aligned} \tag{3.65}$$

$$s_{\rm II}(x) = x + \phi_{\rm II} + 2k_{\rm II}\ln(x) \ , \tag{3.66}$$

where $L_{\rm II}$, $M_{\rm II}$ are unknown constants. The parameters are now[1]

$$\begin{aligned} k_{\rm II} &= \frac{1}{2\pi}\ln(1-\xi) \ , \\ e^{2\imath\phi_{\rm II}} &= 2^{4\imath k_{\rm II}}\frac{\Gamma^2(-\imath k_{\rm II})}{\Gamma^2(\imath k_{\rm II})} \ . \end{aligned} \tag{3.67}$$

Only for $\lambda = 0$, $\xi \in (1, \infty)$ and case I applies. As mentioned above for $\lambda = 0$ the extra particle's Green's function becomes identical with the single particle Green's function of a system of impenetrable Bosons. For that case Vaidya and Tracy [74, 75] derived an asymptotic expansion, whose first terms read

$$G_{\rm I}(x) = \frac{G_\infty}{\sqrt{x}}\left\{1 + \frac{1}{8x^2}\left[\cos(2x) - \frac{1}{4}\right] + \mathcal{O}(x^{-3})\right\} \ . \tag{3.68}$$

[1]There is a typo in Eq. (1.29c) of Ref. [85]. The square bracket has to read $[\theta + \frac{A^2}{4} - n^2 - 1]$.

This result can not completely be derived from Eqs. (3.62) and (3.63). The constant $G_\infty \approx \pi\sqrt{e}2^{-1/3}A^{-6}$, where $A \approx 1.2824$ is Glaisher's constant, cannot be obtained from the asymptotic expansion of $\sigma_V$. Neither can the constant term in the square bracket. Rather Eq. (3.68) fixes the constant $M_{\rm I} = 2^{-4}$.

For $|\lambda| > 0$ case II applies. We write $k_{\rm II} = \imath\alpha$, where

$$\alpha = \frac{\mathrm{sgn}(\lambda)}{2} - \frac{1}{\pi}\arctan(\lambda) = \frac{\mathrm{sgn}(\hat{K}) - \hat{K}}{2} \tag{3.69}$$

is a number between $-1/2$ and $1/2$. The cases $\alpha = 0$ and $\alpha \neq 0$ have to be treated separately. By expanding Eq. (3.66) for $\alpha = 0$ up to leading order we obtain $\sigma_V \sim x\cot(x) - 1$. This together with $\hat{K} = 1$ yields

$$G_{\rm I}(x) \;=\; e^{\imath x}\frac{\sin(x)}{x}\;. \tag{3.70}$$

This is the well known Green's function of a system of free Fermions. Likewise, it could have been derived from Eq. (3.47) in the limit $\lambda \to \infty$. It is easily checked that $x\cot(x) - 1$ is an exact solution to the Painlevé V differential equation (3.58) to the boundary condition $\lim_{x\to 0}\sigma_V = -x^2/3$.

For $\alpha \neq 0$ the asymptotic expansion yields

$$\begin{aligned} G_{\rm I}(x) \;=\; G_\infty^{(\lambda)} x^{-(\hat{K}^2+1)/2}\Bigg[1 - \frac{\Gamma^2(\alpha)}{\Gamma^2(-\alpha)}\frac{e^{2\imath x}}{(2x)^{4|\alpha|}} - \frac{M_{\rm II} - \imath(1-3\alpha^2)}{x} \\ +\mathcal{O}\left(x^{-4|\alpha|-\min(4|\alpha|,1)}\right)\Bigg]\;. \end{aligned} \tag{3.71}$$

The exponent of the leading order term can be written as a function of the extra particle's momentum $\hat{K}$. It interpolates between $-1/2$ for the extra particle's momentum in the core of the Fermi-sea ($\hat{K} = 0$) and $-1$ for the extra particle's momentum right at the edge of the Fermi-sea ($\hat{K} = \pm 1$). As $\alpha$ approaches zero, respectively $\hat{K}$ the Fermi-edge, the approximation in Eq. (3.71) becomes poorer and poorer. Finally for $\alpha = 0$ infinitely many terms contribute to the same order and sum up to the simple result (3.70). Within the present approach the constants $G_\infty^{(\lambda)}$ and $M_{\rm II}$ of the asymptotic expansion can not be determined. The first sub-leading term in the expansion is of order $\min(4|\alpha|, 1)$.

In Fig. 3.7, we show the plot of $G_{\rm I}(y^-)$ for different values of $\hat{K}$. The full lines are obtained from the representation of $G_{\rm I}(y^-)$ as Töplitz determinant (see Eq. (A.43) in Appendix A) for $N = 29$ sea particles. This is compared with the result obtained from the asymptotic expansion in the thermodynamic limit as given in Eqs. (3.59), (3.68) and (3.71). The constants $G_\infty^{(1,1)} \approx 0.6$ and $M_{\rm II} \approx 0$ were numerically approximated. In particular for the non-oscillatory curve with $\hat{K} = 0$ the asymptotic expansion (3.71) works remarkably well almost everywhere.

### 3.3.3 Finite interaction strength

An analytical evaluation of the Green's function in the thermodynamic limit for finite interaction strength seems momentarily not to be in reach. Nevertheless the representation (3.46) of $G_{\rm I}(y^-)$ allows us to deduce the short distance behavior. To that end,

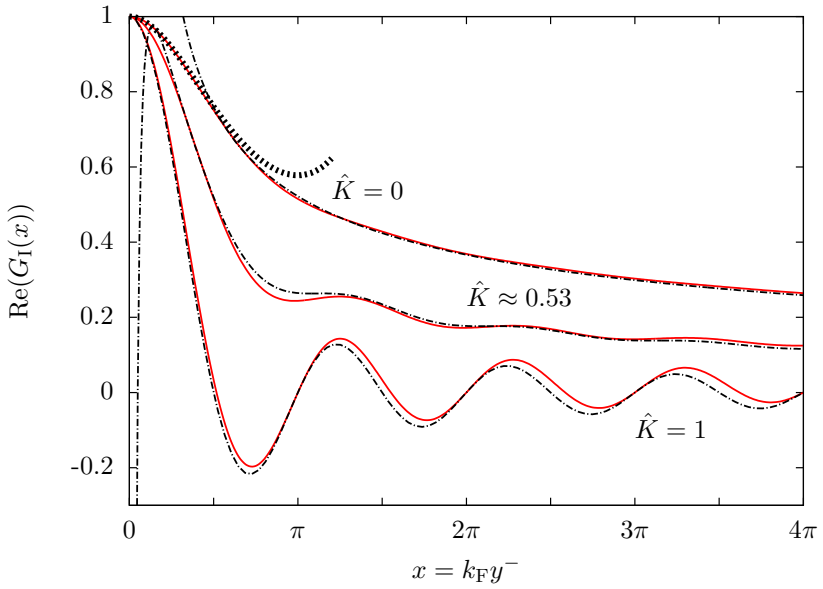


Figure 3.7: Comparison of the asymptotic expansion (dot-dashed lines, black) as given in Eqs. (3.68), (3.70) and (3.71) with the numerical results (full lines, red) obtained form Eq. (A.43) for $N = 29$ . The values of $\hat{K}$ are $\hat{K} = 0$, $\hat{K} \approx 0.53$ and $\hat{K} = 1$. The thick dotted line (black) corresponds to the small $x$ expansion for $\hat{K} = 0$.

we expand $G_{\rm I}(y^-)$ for small $y^-$. As starting point we take the quantities $g_{nm}$ defined in Eq. (3.44). Expanding them up to the quadratic term yields

$$g_{nm} = (b_n\delta_{nm} - 2c) + \left[\imath(-k_n b_n \delta_{nm} + 2c(k_n + k_m - \Lambda)) - 2c^2\right] y^- \tag{3.72}$$
$$+\frac{1}{4}\left[-2k_n^2 b_n\delta_{nm} + 4c(k_n^2 + k_m^2 + k_n k_m) - 4c(k_n + k_m)(\Lambda - \imath c)\right]\left(y^-\right)^2 + \mathcal{O}\left((y^-)^3\right) \ ,$$

where we have introduced the notation

$$b_n = L[(k_n - \Lambda)^2 + c^2] + 2c \ . \tag{3.73}$$

For the linear combination of the quantities $g_{nm}$ as it appears in Eq. (3.46) we obtain

$$\begin{aligned} & g_{nm} - g_{n+1m} - g_{nm+1} + g_{n+1m+1} \qquad (3.74)\\ = \ & [(b_n + b_{n+1})\delta_{nm} - b_n\delta_{nm+1} - b_{n+1}\delta_{n+1m}] \\ - \ & \imath\left[(b_n k_n + b_{n+1}k_{n+1})\delta_{nm} - b_n k_n \delta_{nm+1} - b_{n+1}k_{n+1}\delta_{n+1m}\right] y^- \\ - \ & \frac{1}{2}\Big( \left[(b_n k_n^2 + b_{n+1}k_{n+1}^2)\delta_{nm} - b_n k_n^2\delta_{nm+1} - b_{n+1}k_{n+1}^2\delta_{n+1m}\right] \\ & -2c(k_n - k_{n+1})(k_m - k_{m+1})\Big)(y^-)^2 + \mathcal{O}((y^-)^3) \ . \end{aligned}$$

We substitute the expansion (3.74) into Eq. (3.46) and write $G_{\rm I}(y^-)$ formally in a power series up to the quadratic term

$$G_{\rm I}(y^-) \ = \ 1 + \imath G_{\rm I}^{(1)} y^- - G_{\rm I}^{(2)}(y^-)^2 + \mathcal{O}\left((y^-)^3\right) \ . \tag{3.75}$$

Our numerical results (see Figs. 3.5 and 3.6) together with the results obtained in the hardcore limit indicate that $G_{\rm I}(y^-)$ is knotless for $|K| < k_{\rm F}$. Therefore the short distances decay of the Green's function is dominated by the length $[G_{\rm I}^{(2)}]^{-1/2}$ to which we refer as correlation length.

In Eq. (3.75), we have used that, by definition, $G_{\rm I}(0) = 1$ is normalized to unity at $y^- = 0$. Comparing with Eq. (3.46) we have

$$P_N \equiv (N!L\,|C_N|^2)^{-1} = \det\left[(b_n + b_{n+1})\delta_{nm} - b_n\delta_{nm+1} - b_{n+1}\delta_{n+1m}\right]_{n,m=1,\ldots,N} \quad , \tag{3.76}$$

where the notation $P_N$ for the determinant on the right hand side has been introduced. It can be evaluated by expansion with respect to the last column. This yields the recursion relation

$$P_N = (b_N + b_{N+1})P_{N-1} - b_N^2 P_{N-2} \tag{3.77}$$

which is easily shown to be solved by

$$P_N = \left(\prod_{j=1}^{N+1} b_j\right) \sum_{j=1}^{N+1} \frac{1}{b_j} \; . \tag{3.78}$$

Equating Eqs. (3.76) and (3.78) yields for the normalization constant

$$|C_N|^{-2} = LN! \left(\prod_{j=1}^{N+1} b_j\right) \sum_{j=1}^{N+1} \frac{1}{b_j} \tag{3.79}$$

and after substituting $b_j$ according to Eq. (3.73) we arrive at the expression (3.4).

Similarly the first and second order coefficient can be evaluated. The calculation is somewhat lengthy but straightforward. It yields

$$G_{\rm I}^{(1)} = \left(\sum_{j=1}^{N+1} \frac{1}{b_j}\right)^{-1} \left(\sum_{j=1}^{N+1} \frac{K - k_j}{b_j}\right), \tag{3.80}$$

$$G_{\rm I}^{(2)} = \frac{1}{2}\left(\sum_{j=1}^{N+1} \frac{1}{b_j}\right)^{-1} \left(\sum_{j=1}^{N+1} \frac{(K-k_j)^2}{b_j} - c\sum_{j=1}^{N+1}\sum_{l=1}^{N+1} \frac{(k_j - k_l)^2}{b_j b_l}\right) \; . \tag{3.81}$$

By definition, the antisymmetric part of $G_{\rm I}(y^-)$ is purely imaginary while the symmetric part is always real. This is reflected in the expansion (3.75) and in the expansion

coefficients (3.80) and (3.81). In the thermodynamic limit they translate to

$$G_{\mathrm{I}}^{(1)} = \frac{c}{v(\Lambda, k_{\mathrm{F}})} \int_{-k_{\mathrm{F}}}^{+k_{\mathrm{F}}} dk \frac{(K-k)}{(k-\Lambda)^2 + c^2} \ , \tag{3.82}$$

$$G_{\mathrm{I}}^{(2)} = \frac{c}{2v(\Lambda, k_{\mathrm{F}})} \left( \int_{-k_{\mathrm{F}}}^{+k_{\mathrm{F}}} dk \frac{(K-k)^2}{(k-\Lambda)^2 + c^2} - \frac{c}{2\pi} \int_{-k_{\mathrm{F}}}^{+k_{\mathrm{F}}} dk \int_{-k_{\mathrm{F}}}^{+k_{\mathrm{F}}} dk' \frac{(k-k')^2}{[(k-\Lambda)^2 + c^2][(k'-\Lambda)^2 + c^2]} \right) . \tag{3.83}$$

The integrals can be done. However, the resulting expressions give little insight. Thus we do not state them here. Equating $G_{\mathrm{I}}^{(1)}$ and $G_{\mathrm{I}}^{(2)}$ in the hardcore limit yields

$$\lim_{c\to\infty} G_{\mathrm{I}}^{(1)} = k_{\mathrm{F}} \hat{K} \qquad , \qquad \lim_{c\to\infty} G_{\mathrm{I}}^{(2)} = \frac{k_{\mathrm{F}}^2}{6}(1 + 3\hat{K}^2) \ . \tag{3.84}$$

Note that Eq. (3.84) is in perfect agreement with the results of the Painlevé transcendental evaluation of $G_{\mathrm{I}}(y^-)$ in the hardcore limit, see Eqs. (3.59) and (3.60). On the other hand we obtain in the limit of vanishing interaction strength

$$\lim_{c\to 0} G_{\mathrm{I}}^{(1)} = \lim_{c\to 0} G_{\mathrm{I}}^{(2)} = 0 \ , \tag{3.85}$$

where $\Lambda|_{c=0} = K$ has been used. According to the equation above we have $G_{\mathrm{I}}(y^-) = 1$ for $c = 0$. This is reasonable since then the distinguishable particle does not feel the presence of the Fermi-sea. Great simplification arises for the ground state, where $K = \Lambda = 0$. As obvious from Eq. (3.82) the linear term then vanishes identically and coefficient of the quadratic term yields for the correlation length

$$[G_{\mathrm{I}}^{(2)}]^{-1/2} = \frac{1}{\sqrt{2}k_{\mathrm{F}}} \left[ \frac{c}{2\pi k_{\mathrm{F}}} \left( \frac{\pi}{\arctan(k_{\mathrm{F}}/c)} - 2 \right) \left( 1 - \frac{c}{k_{\mathrm{F}}} \arctan\left( \frac{k_{\mathrm{F}}}{c} \right) \right) \right]^{-1/2} . \tag{3.86}$$

Figure 3.8 shows the correlation length as function of the interaction strength for different values of the extra particle's momentum. In general the correlation length decays monotonically between the values determined by Eqs. (3.85) and (3.84) as $c$ varies from zero to $\infty$. Comparing $[G_{\mathrm{I}}^{(2)}]^{-1/2}$ for the ground state and states with $K \neq 0$, it is found that the closer $K$ is located to the Fermi-edge, the faster is the decay.

For finite interaction strength a further analytical evaluation of the Green's function seems to be difficult. However, the behavior of the Green's function towards the thermodynamic limit might be extracted from Eq. (3.46) by evaluating it numerically for a large number of particles. To that end we choose the quantum numbers $m_j$ as in Eq. (3.9) and measure $y^-$ and $c$ on the scale of $k_{\mathrm{F}}$. Figure 3.9 shows the corresponding plot of the interaction part of the Green's function for $\hat{K} = 0$ and for an excited state

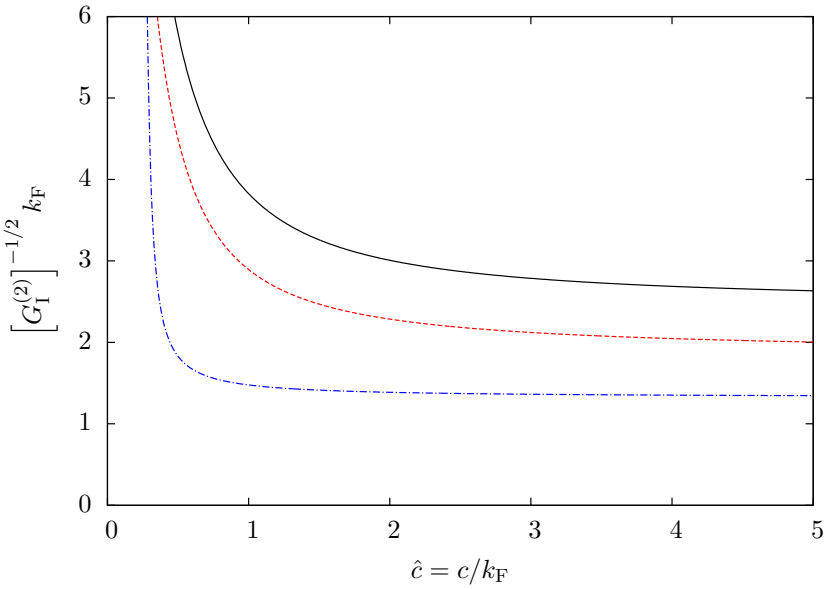


Figure 3.8: Correlation length $[G_{\rm I}^{(2)}]^{-1/2}$ as function of $\hat{c}$ for different values of $\hat{K} = K/k_{\rm F}$. The values for $\hat{K}$ are: $\hat{K} = 0$ (solid line, black), $\hat{K} = 0.5$ (dashed line, red) and $\hat{K} = 0.9$ (dot-dashed line, blue).

with $\hat{K} = 0.6$ for different values of the interaction strength. The particle number is $N = 29$. It is seen that, in accordance with the discussion for the correlation length, the short distance decay of $G_{\rm I}(x)$ is considerably faster for excited states.

## 3.4 Density-density correlation function

The equal time density-density correlation function for a particle of the Fermi-sea and the distinguishable particle has been calculated in Ref. [4]. However, there the results are restricted to the case $K = \Lambda = 0$ i.e. to the ground state. Here we give a brief derivation of the corresponding quantity where the distinguishable particle may occupy an arbitrary exited state within the Fermi-sea.

The density-density correlation function is defined by the $(N-1)$-fold integral

$$R(x,y) \;=\; N \int_0^L dx_1 \cdots \int_0^L dx_{N-1} \left|\Psi(x_1,\ldots,x_{N-1},x,y)\right|^2 . \tag{3.87}$$

Using the wave function (3.2) and the identity (3.12), the expression above is after some algebra rewritten as

$$R(z) \;\propto\; \sum_{n=1}^{N+1} \frac{N}{\left|A_n(1)\right|^2} - \sum_{n\neq m}^{N+1} \frac{e^{\imath(k_n-k_m)z}}{A_n(z)A_m^*(z)} + \mathcal{O}\left(N^{-1}\right) \tag{3.88}$$

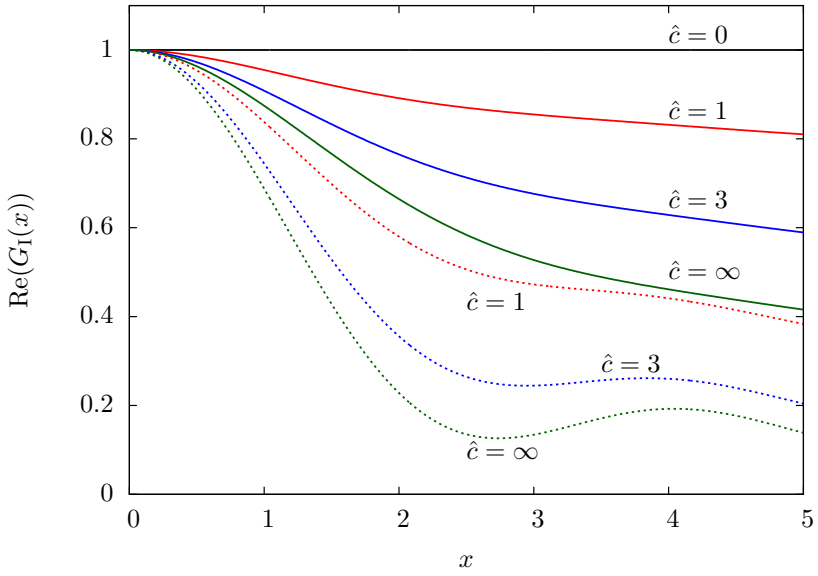


Figure 3.9: Real part of $G_{\mathrm{I}}(x)$ as function of $x = k_F y^-$ for the ground state $\hat{K} = 0$ (solid lines) and for the state with $\hat{K} = 0.6$ (dotted lines). The interaction strength $\hat{c} = c/k_{\mathrm{F}}$ has the values $\hat{c} = 0$ (black), $\hat{c} = 1$ (red), $\hat{c} = 3$ (blue) and $\hat{c} = \infty$ (green). The plots are generated for $N = 29$ sea particles.

with $z = x - y$. In order to evaluate the integrals in Eq. (3.87) we employed the identity (3.12) as well as the relation (3.32). Taking into account the normalization condition

$$\int_0^L dx \int_0^L dy R(x,y) = N \tag{3.89}$$

we obtain

$$R(z) = \frac{\rho}{L}\left(1 - \frac{1}{N\sum_{n=1}^{N+1}\left|A_n(1)\right|^{-2}} \sum_{n,m}^{N+1} \frac{e^{\imath(k_n - k_m)z}}{A_n(z)A_m^*(z)}\right) + \mathcal{O}(N^{-1}) \ . \tag{3.90}$$

In the thermodynamic limit the summations can be replaced by integrations using the density of states (3.11). Then by introducing the dimensionless quantities

$$\hat{c} = c/k_F \quad , \quad \hat{\Lambda} = \Lambda/k_F \quad , \quad \hat{z} = k_F z \quad , \quad \hat{R}(z) = \frac{L\pi}{k_{\mathrm{F}}} R(z) \tag{3.91}$$

the density-density correlation function acquires in leading order of the system size the

form

$$\hat{R}(\hat{z}) = 1 - \frac{\hat{c}}{2v(\hat{\Lambda},1)} \left| \int_{-1}^{+1} dk \frac{e^{\imath k\hat{z}}}{i(k-\hat{\Lambda})+\hat{c}} \right|^2 + \mathcal{O}(N^{-1}) \ . \tag{3.92}$$

We discuss Eq. (3.92) in three limits. First, it is easily seen that in the limit of infinite strong interaction strength the density-density correlation function becomes identical with that of free Fermions i.e.

$$\lim_{\hat{c}\to\infty} \hat{R}(\hat{z}) = 1 - \left(\frac{\sin\hat{z}}{\hat{z}}\right)^2 . \tag{3.93}$$

On the other hand, the second term in Eq. (3.92) vanishes like $c$ as $\hat{c} \to 0$ and hence it follows that

$$\lim_{\hat{c}\to 0} \hat{R}(\hat{z}) = 1 \ . \tag{3.94}$$

Finally we consider the local density-density correlation function i.e. the limit $\hat{z} \to 0$. This is the density of the sea-Fermions at the position of the distinguishable particle. According to Eqs. (3.93) and (3.94) we expect a transition from $\hat{R}(0) = 1$ to $\hat{R}(0) = 0$ as $\hat{c}$ increases from zero to $+\infty$. The integral in Eq. (3.92) can be evaluated and the resulting expression reads

$$\hat{R}(0) = 1 - \frac{\hat{c}}{2v(\hat{\Lambda},1)} \left| \ln\left(\frac{\hat{c}+\imath(1-\hat{\Lambda})}{\hat{c}-\imath(1+\hat{\Lambda})}\right) \right|^2 . \tag{3.95}$$

As outpointed in Sec. 3.2.2, the equation above directly relates to the mean interaction energy via Eq. (3.27) and allows to calculate the latter quantity for exited states i.e. $\hat{K} \neq 0$.

In Fig. 3.4, we show the plot of $\hat{R}(\hat{z})$ as function of the distance $\hat{z}$ and the interaction strength $\hat{c}$. It is seen that for small distances the density-density correlation function decays monotonically from unity to zero as $\hat{c}$ increases. This behavior changes at larger distances and $\hat{R}(\hat{z})$ exhibits a local minimum as function of $\hat{c}$. The corresponding plot for the ground state is qualitatively similar to that of Fig. 3.4.

## 3.5 Survival probability and local density of states

So far our considerations have been restricted to stationary quantities. The aim of the present section is to push forward an approach to study time dependent properties as well. In the course of doing so we restrict the discussion to the survival probability and the local density of states. Their definition will be stated below.

The first thing need to be done in order to discuss dynamical quantities, is to fix the initial state of the system. Depending on the physical situation to be described, there are several possibilities. We will consider the following scenario: For times $t \leq 0$

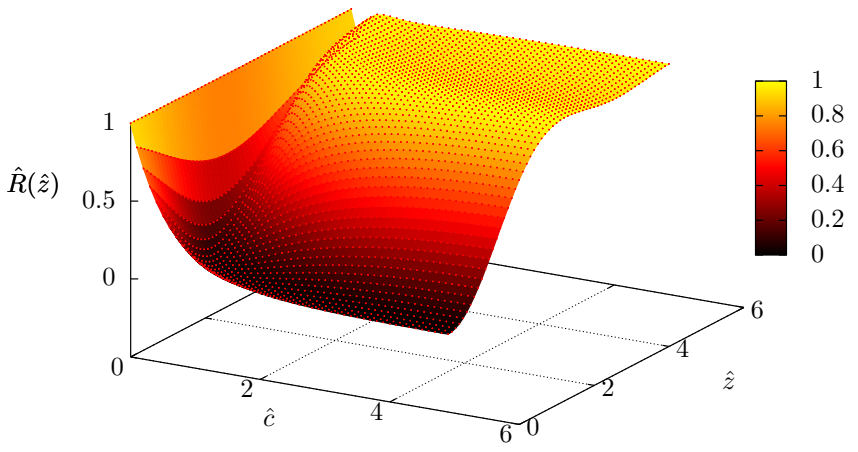


Figure 3.10: Density-density correlation function $\hat{R}(\hat{z})$ as function of the interaction strength $\hat{c} = c/k_{\mathrm{F}}$ and distance $\hat{z} = zk_{\mathrm{F}}$ for the exited state with $\hat{K} = 0.6$.

we assume the interaction in the Hamiltonian (3.1) to be switched off, that is we set $c = 0$. The system then constitutes of $N$ identical (spin-down) Fermions and a single distinguishable particle with no interaction among them. This fixes our initial state $|\chi_0\rangle$. For periodic boundary conditions the corresponding wave function reads in coordinate representation

$$\chi_0(\mathbf{x}, \mathbf{k}, y, q_y) = \frac{e^{\imath q_y y}}{\sqrt{N! L^{N+1}}} \det\left[e^{\imath q_j x_l}\right]_{j,l=1,\ldots,N}, \tag{3.96}$$

where the constant in the denominator ensures the normalization to unity. To complete the description of our initial state, the quasi-momenta $q_y$ and $q_l$, $l = 1, \ldots, N$ need to be fixed. We assume the spin-down particles to be in the ground state, whereas the distinguishable particle is free to occupy an arbitrary state (inside or outside the Fermi-sea). Thus the quasi-momenta of the non-interacting system are

$$\begin{aligned} q_y &= \frac{2\pi}{L} n_y \quad , \quad n_y \in \mathbb{Z} \\ q_l &= \frac{2\pi}{L} n_l \quad , \quad n_l = \left(-\frac{N+1}{2} + l\right) \quad , \quad l = 1, \ldots, N \quad . \end{aligned} \tag{3.97}$$

Since the wave function in Eq. (3.96) is characterized by the quantum numbers $n_y$ and $n_l$, we denote the corresponding state also by $|\chi_0(n_y, \mathbf{n})\rangle$. By Eqs. (3.96) and (3.97) the description of the system for $t \leq 0$ is completed.

Now, for times $t > 0$, we assume the interaction to be switched on and thus the system is described by the full Hamiltonian as given in Eq. (3.1). The wave function

$|\chi(n_y, \mathbf{n}, t)\rangle$ for times $t > 0$ is obtained from the initial state $|\chi_0(n_y, \mathbf{n})\rangle$ through evolution with the Hamiltonian $H$ according to

$$|\chi(n_y, \mathbf{n}, t)\rangle = e^{-\imath Ht}|\chi_0(n_y, \mathbf{n})\rangle \ . \tag{3.98}$$

Obviously a non-trivial time evolution is provided only if the system initially is described by a state which is not an eigenfunction of $H$. As is readily seen this is the case if the initial state is determined by Eq. (3.96).

We consider the survival probability amplitude $S(t)$. It is defined as the overlap of the wave function for $t \leq 0$ with the one for $t > 0$, that is

$$S(t) = \langle\chi_0(\mathbf{n}, n_y)|\chi(\mathbf{n}, n_y, t)\rangle = \langle\chi_0(\mathbf{n}, n_y)|e^{-\imath Ht}|\chi_0(\mathbf{n}, n_y)\rangle \ . \tag{3.99}$$

The square of the modulus of $S(t)$ measures the probability to find the system after a time $t$ in the initial state. Due to its probabilistic interpretation it is bounded by unity. According to the definition (3.99) we have $S(0) = 1$.

One possibility to further evaluate the expression (3.99) is to use the eigenstates of $H$. The corresponding wave function is in coordinate representation given by Eq. (3.2). For periodic boundary conditions the eigenstates are characterized by the quantum numbers $m_j,\ j = 1, \ldots, N+1$ and the quantum number $J$. Thus we refer to them also by $|\Psi(\mathbf{m}, J)\rangle$. Inserting a complete set of eigenstates of $H$, the survival probability amplitude acquires the form

$$S(t) = \frac{1}{(N+1)!} \sum_{\{\mathbf{m},J\}} |\langle\chi_0(\mathbf{n}, n_y)|\Psi(\mathbf{m}, J)\rangle|^2 e^{-\imath E(\mathbf{k})t} \ , \tag{3.100}$$

where

$$E(\mathbf{k}) = \sum_{j=1}^{N+1} k_j^2 \tag{3.101}$$

is the energy associated with the eigenstate $|\Psi(\mathbf{m}, J)\rangle$. In Eq. (3.100), the quantum numbers $\{m_j\}_{j=1,\ldots,N+1}$ have to be summed independently from $-\infty$ to $+\infty$. The sum over the quantum number $J$ has to be taken in between $\pm(N+1)/2$. The factor $(N+1)!$ in the denominator divides out the number of configurations that are equivalent due to the antisymmetry of $|\Psi(\mathbf{m}, J)\rangle$.

A quantity that closely relates to $S(t)$ is the local density of states. It is defined via

$$L(E) = \frac{1}{(N+1)!} \sum_{\{\mathbf{m},J\}} |\langle\chi_0(\mathbf{n}, n_y)|\Psi(\mathbf{m}, J)\rangle|^2 \delta(E - E(\mathbf{k})) \tag{3.102}$$

and normalized to unity

$$\int_{-\infty}^{+\infty} dE L(E) = 1 \ . \tag{3.103}$$

From Eqs. (3.100) and (3.102) we find that both quantities relate to each other via Fourier transformation i.e.

$$L(E) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \; S(t) e^{\imath E t} \; . \tag{3.104}$$

In the following we further evaluate $|S(t)|^2$ by using perturbation theory.

### 3.5.1 Perturbative approach

Switching on the interaction at $t = 0$ will change the initial state. Consequently the probability to find the system in the initial state for times $t > 0$ is expected to decay. A standard approximation scheme to evaluate $|S(t)|^2$ for small values of the interaction strength is to use perturbation theory. The survival probability might be written as

$$|S(t)|^2 \approx e^{-\Gamma t} \; , \tag{3.105}$$

where $\Gamma$ denotes the decay rate per unit time. According to Fermis-golden rule [87] it can be approximated by

$$\Gamma = 2\pi \sum_{\{|f\rangle\}} |\langle f|\hat{V}|\chi_0(\mathbf{n}, n_y)\rangle|^2 \delta(E_f - E_0) \; . \tag{3.106}$$

Here $\{|f\rangle\}$ denotes the set of states into the initial state can decay due to the interaction. Furthermore $E_0 = E_\mathrm{F} + q_y^2$ and $E_f$ denote the energies of the states $|\chi_0(\mathbf{n}, n_y)\rangle$ and $|f\rangle$, respectively. The $\delta$-function in Eq. (3.106) indicates that the energy is conserved. To evaluate the expression above, we write the interaction potential in second quantization

$$\hat{V} = \frac{c}{L} \sum_k \sum_{k'} \sum_{k''} b_{k'}^\dagger b_{k''} a_k^\dagger a_{k+k'-k''} \; , \tag{3.107}$$

where $b_k^{(\dagger)}$ and $a_k^{(\dagger)}$ are the annihilation (creation) operators for the distinguishable particle and the particles of the Fermi-sea, respectively. They fulfill the relations

$$[b_k^\dagger, b_{k'}^\dagger] = [b_k, b_{k'}] = 0 \quad , \quad [b_k, b_{k'}^\dagger] = \delta_{k,k'} \; , \tag{3.108}$$

$$\{a_k^\dagger, a_{k'}^\dagger\} = \{a_k, a_{k'}\} = 0 \quad , \quad \{a_k, a_{k'}^\dagger\} = \delta_{k,k'} \tag{3.109}$$

and commute among each other. We note that the potential supports momentum conservation. Acting with the interaction potential (3.107) onto the initial state causes particle hole excitations. The final states $|f\rangle$ are thus fully characterized by the momentum of the created particle $k_p$, the created hole $k_h$ and the new momentum $q_y'$ of the distinguishable particle. The energy is therefore $E_f = E_\mathrm{F} + k_p^2 + q_y'^2 - k_h^2$ and we write the corresponding states as $|f\rangle = |q_y', k_p, k_h\rangle$. For the matrix elements in Eq. (3.106) we

obtain

$$\begin{aligned}\langle f|V|\chi_0(\mathbf{n},n_y)\rangle &= \frac{c}{L}\sum_k\sum_{k'}\sum_{k''}\langle k'_y,k_p,k_h|b^\dagger_{k'}b_{k''}a^\dagger_k a_{k+k'-k''}|\chi_0(\mathbf{n},n_y)\rangle \quad (3.110)\\ &= \frac{c}{L}\sum_k\sum_{k'}\sum_{k''}\delta_{k_h,k+k'-k''}\delta_{k_p,k}\delta_{k_y,k''}\delta_{k',k'_y}\\ &= \frac{c}{L}\delta_{k_h,k_p+k'_y-k_y}\end{aligned}$$

and hence the decay rate $\Gamma$ acquires the form

$$\begin{aligned}\Gamma &= \frac{2\pi c^2}{L^2}\sum_{k_p}\sum_{k_h}\sum_{q'_y}\delta_{k_h-k_p-q'_y+q_y}\delta(k_p^2+q_y'^2-k_h^2-q_y^2) \quad (3.111)\\ &= \frac{2\pi c^2}{L^2}\sum_{k_p}\sum_{k_h}\delta\Big(2(k_p-q_y)(k_p-k_h)\Big)\ .\end{aligned}$$

Since we always have $|k_p|>k_{\rm F}>|k_h|$ it follows that $k_p-k_h\neq 0$. Hence $\Gamma$ is only non-zero if $q_y=k_p$ and since $|k_p|>k_F$, the initial state decays only if the extra particle's momentum lies outside the Fermi-sea. The reason for this is that for $|q_y|<k_{\rm F}$ there are no particle-hole excitations which preserve the total energy and momentum. In other words: the initial state is for $|q_y|<k_{\rm F}$ non-degenerated. On the other hand for $|q_y|>k_{\rm F}$ the initial state is degenerated and there exist particle-hole excitations for which energy and momentum are conserved.

We evaluate the expression Eq. (3.111) in the thermodynamic limit by using

$$\delta(2(k_p-k_h)(k_p-q_y))=\frac{\delta(k_p-k_h)}{2|k_p-q_y|}+\frac{\delta(k_p-q_y)}{2|q_y-k_h|}\ . \quad (3.112)$$

Since we have $k_p\neq k_h$, the first term in the equation above does not contribute and the decay rate acquires the form

$$\begin{aligned}\Gamma &= \frac{c^2}{4\pi}\Theta(|q_y|-k_{\rm F})\int_{-k_{\rm F}}^{k_{\rm F}}\frac{dk_h}{|q_y-k_h|}\int_{\mathbb{R}/[-k_{\rm F},k_{\rm F}]}dk_p\delta(k_p-q_y)\\ &= -\frac{c^2}{4\pi}\Theta(|q_y|-k_{\rm F})\ln\left(\frac{|q_y|-k_{\rm F}}{|q_y|+k_{\rm F}}\right)\ . \quad (3.113)\end{aligned}$$

Substitution into Eq. (3.105) yields for the survival probability

$$|S(t)|^2=\begin{cases}1 & \text{for} \quad |q_y|<k_{\rm F}\\ \left(\dfrac{|q_y|-k_{\rm F}}{|q_y|+k_{\rm F}}\right)^{c^2t/(4\pi)} & \text{for} \quad |q_y|>k_{\rm F}\end{cases}\ . \quad (3.114)$$

Hence the decay of the initial state crucially depends on the momentum of the extra particle. Within the present approach this might be understood by the argument given above. However, in general on should expect a decay also for the case that $|q_y|<k_{\rm F}$, since the initial state is not an eigenstate of the Hamiltonian.

### 3.5.2 Towards a non-perturbative evaluation

The failure of the perturbative approach indicates that there is a crucial difference between the cases $|q_y| < k_{\rm F}$ and $|q_y| > k_{\rm F}$. Here we present steps which might lead to a non-perturbative evaluation. We restrict the discussion to the local density of states. The survival probability amplitude can be obtained by Fourier transformation.

According to Eq. (3.102), the main ingredient for the local density of states is the overlap of the initial state with the eigenstates of the Hamiltonian $H$. In coordinate representation the overlap integral reads

$$\langle\chi_0(\mathbf{n},n_y)|\Psi(\mathbf{m},J)\rangle = \int_0^L dy \int_0^L dx_1 \cdots \int_0^L dx_N \chi^*(\mathbf{x},\mathbf{k},y,q_y)\Psi(\mathbf{x},\mathbf{k},y,\Lambda) \ . \quad (3.115)$$

To evaluate it, we use the explicit expressions (3.2) and (3.96) for the wave functions as well as the relation (3.12). Then employing translational invariance we find

$$\begin{aligned}&\langle\chi_0(\mathbf{n},n_y)|\Psi(\mathbf{m},J)\rangle \quad (3.116)\\ &= \sqrt{\frac{N!}{L^{N-1}}} C_N \delta_{K,q_y} \sum_{n=1}^{N+1} (-1)^{n+N+1} \prod_{j\neq n}^{N+1} A_j(\Lambda,1) \det\left[\int_0^L dx e^{\imath(k_j-q_l)x}\right]_{\substack{j=1,\ldots,N+1\neq n\\ l=1,\ldots,N}} ,\end{aligned}$$

where the Kronecker-$\delta$ indicates that the total momentum $q_y$ of the initial state is conserved as the interaction is switched on. To further evaluate the expression (3.116), we assume $c > 0$ and $|q_y| < k_{\rm F}$. The evaluation of the integral in Eq. (3.116) then yields

$$\int_0^L dx e^{\imath(k_j-q_l)x} = \frac{e^{\imath k_j L}-1}{\imath(k_j-q_l)} = \frac{2\imath c}{A_j(\Lambda,1)(k_j-q_l)} \ , \quad (3.117)$$

where the Bethe-Ansatz equation (3.4) has been used. Substituting into Eq. (3.116) we have

$$\langle\chi_0(\mathbf{n},n_y)|\Psi(\mathbf{m},J)\rangle = \sqrt{\frac{N!}{L^{N-1}}} C_N (2\imath c)^N \delta_{K,q_y} \det\left[\frac{1}{k_j-q_l}\middle| 1\right]_{\substack{j=1,\ldots,N+1\\ l=1,\ldots,N}} . \quad (3.118)$$

Now the determinant on the right hand side is identified as a special case of Cauchy's determinant

$$\det\left[\frac{1}{x_j-y_l}\right]_{j,l=1,\ldots,N} = \frac{\prod_{j<l}^N (x_j-x_l)\prod_{j<l}^N (y_j-y_l)}{\prod_{j,l=1}^N (x_j-y_l)} \ . \quad (3.119)$$

Applied to Eq. (3.118), this yields for the squared modulus of the overlap integral

$$|\langle\chi_0(\mathbf{n},n_y)|\Psi(\mathbf{m},J)\rangle|^2 = \frac{(2c)^{2N}N!|C_N|^2}{L^{N-1}} \frac{\prod_{j<l}^{N+1}(k_j-k_l)^2 \prod_{j<l}^N (q_j-q_l)^2}{\prod_{j=1}^{N+1}\prod_{l=1}^N (k_j-q_l)^2} \delta_{K,q_y} \ , \quad (3.120)$$

which holds for $c > 0$ and $|q_y| < k_{\rm F}$. Substituting the expression above into the definition (3.102) of $L(E)$ we face the problem to carry out the summations over all quantum-numbers. This is a serious obstacle for a further analytical evaluation of $L(E)$. It might be treated as follows:

Our first goal is to replace the summations over all quantum numbers by integrals. This can be achieved via the Euler-MacLaurin formula [88] which states a connection between finite sums and integrals. It reads

$$\sum_{m=u}^{v} f(m) \propto \int_{u}^{v} dm f(m) + \frac{f(v)+f(u)}{2} + \frac{f'(v)-f'(u)}{12} + \cdots \ , \tag{3.121}$$

where $u$ and $v$ are integers. Demanding $f$ and all of its derivatives to vanish at $\pm\infty$, the sum on the left hand side can for large values of $u, v$ be approximated by the corresponding integral. Assuming this to be the case in the present context, the summations in Eq. (3.102) might be replaced by integrals according to

$$\sum_{J=-\frac{N+1}{2}}^{\frac{N+1}{2}} \sum_{m_1=-\infty}^{+\infty} \cdots \sum_{m_{N+1}=-\infty}^{+\infty} \longrightarrow \int_{-\infty}^{+\infty} dm_1 \cdots \int_{-\infty}^{+\infty} dm_{N+1} \int_{-\frac{N+1}{2}}^{+\frac{N+1}{2}} dJ \ . \tag{3.122}$$

However, the expression (3.120) is a function of the quasi-momenta $\mathbf{k}$ and the parameter $\Lambda$. Therefore we should make a change of coordinates from the set of quantum numbers $\{m_j, J\}_{j=1,\ldots,N+1}$ as independent variables to the quasi-momenta $\mathbf{k}$ and the parameter $\Lambda$. The Jacobian of the corresponding coordinate transformation can be evaluated. Using the Bethe-Ansatz equations (3.4) and (3.5) we find

$$\left| \frac{\partial(m_1, \ldots, m_{N+1}, J)}{\partial(k_1, \ldots, k_{N+1}, \Lambda)} \right| = \frac{2c|C_N|^2}{(2\pi)^{N+1} N!} \prod_{j=1}^{N+1} \frac{1}{(k_j - \Lambda)^2 + c^2} \ . \tag{3.123}$$

Collecting everything, the local density of states can by means of Eqs. (3.120), (3.122) and (3.123) be written as

$$L(E) \propto \int_{-\infty}^{+\infty} \frac{d\Lambda}{c} \int_{-\infty}^{+\infty} dk_1 \cdots \int_{-\infty}^{+\infty} dk_{N+1} \delta_{K,q_y} \delta(E - E(\mathbf{k})) e^{-\bar{A}(\mathbf{k},\mathbf{q})} \ , \tag{3.124}$$

where we have introduced

$$\begin{aligned} \bar{A}(\mathbf{k},\mathbf{q}) &= 2 \sum_{j=1}^{N+1} \sum_{l=1}^{N} \ln \left| \frac{k_j - q_l}{k_{\rm F}} \right| - \sum_{j \neq l}^{N+1} \ln \left| \frac{k_j - k_l}{k_{\rm F}} \right| - \sum_{j \neq l}^{N} \ln \left| \frac{q_j - q_l}{k_{\rm F}} \right| \\ &+ \sum_{j=1}^{N+1} \ln \left| \frac{(k_j - \Lambda)^2 + c^2}{c^2} \right| \ . \end{aligned} \tag{3.125}$$

The form (3.124) may serves as starting point for a further analytical evaluation of $L(E)$. In the thermodynamic limit the multiple integral over the quasi-momenta might be treated as path integral.

### 3.5.3 Overlap in the hardcore limit

We study the overlap between the interacting and the non-interacting wave function for the case of infinite strong interaction strength. The overlap between the two wave functions is relevant in the context of a phenomena called the *orthogonality catastrophe* or *infrared catastrophe*. This term was introduced by Anderson [89] and is usually defined as follows [90]: Consider the wave function of a many body system immediately after the injection of an impurity particle and the wave function of the ground state (with impurity). The vanishing of the corresponding overlap integral in the thermodynamic limit is what is called the orthogonality catastrophe. In particular it has been related to the non Fermi-liquid behavior of electron systems [90].

Our scenario of instantaneously switching on the interaction between the extra particle and the particle's of the Fermi-sea might be considered as the injection of an impurity particle. The non-interacting wave function describes the system immediately after the injection, whereas the interacting wave function determines the ground state in the presence of the impurity. In the following we evaluate the overlap in the hardcore limit for the case that the extra particle's momentum is located within the Fermi-sea.

From the Bethe-Ansatz equations (3.4) and (3.5) we obtain in the hardcore limit

$$k_j = \frac{2\pi}{L}\left(m_j + \beta\right) \quad , \quad \lambda = \tan\left(\pi\left(\beta - \frac{1}{2}\right)\right) \quad , \quad \beta = \frac{1}{2} - \frac{J}{N+1} \; . \tag{3.126}$$

Here $m_j \in \mathbb{Z}$, $j = 1, \ldots, N+1$ are the quantum numbers of the interacting system and $\beta \in [0,1]$. Substituting these expressions into Eq. (3.120) we obtain

$$|\langle \chi_0(\mathbf{n}, n_y)|\Psi(\mathbf{m}, J)\rangle|^2 = \frac{\delta_{K,q_y}}{N+1}\left(\frac{\sin(\pi\beta)}{\pi}\right)^{2N} \frac{\prod_{j<l}^{N+1}(m_j - m_l)^2 \prod_{j<l}^{N}(n_j - n_l)^2}{\prod_{j=1}^{N+1}\prod_{l=1}^{N}(m_j - n_l + \beta)^2} \; , \tag{3.127}$$

where $n_l$ are the quantum numbers of the non-interacting system. We emphasize that the expression above is well defined for all values of $\beta \in [0,1]$. To evaluate it further, we consider the ground state configuration

$$\mathbf{n} = \left\{-\frac{N+1}{2} + l\right\}_{l=1,\ldots,N} \quad , \quad \mathbf{m} = \left\{-\frac{N+3}{2} + l\right\}_{l=1,\ldots,N+1} \; . \tag{3.128}$$

For this particular choice of the quantum numbers the extra particle's momentum $q_y = 2\pi n_y/L$ directly relates to $\beta$. The momentum conservation $\delta_{K,q_y}$ yields

$$q_y = \sum_{j=1}^{N+1} k_j \qquad \Longleftrightarrow \qquad n_y = \frac{N+1}{2}(2\beta - 1) \; . \tag{3.129}$$

Thus, when varying $\beta$ from zero to unity the quantum number of the extra particle varies in between $\mp(N+1)/2$. For the ground state we have $n_y = 0$ and therefore $\beta = 1/2$.

We aim at evaluating Eq. (3.127) in the large $N$ limit. To that end we express the products in the nominator in terms of Γ-functions

$$\prod_{j<l}^{N}(n_j-n_l)^2=\prod_{j=1}^{N-1}\Gamma^2(j+1) \quad , \quad \prod_{j<l}^{N+1}(m_j-m_l)^2=\prod_{j=1}^{N}\Gamma^2(j+1)\ . \tag{3.130}$$

The evaluation of the double product in the denominator is more lengthy but straightforward as well. Separating the terms for which the difference $m_j-n_l$ becomes zero or unity, this expression can be written as

$$\begin{aligned}\prod_{j=1}^{N+1}\prod_{l=1}^{N}(m_j-n_l+\beta)^2 &= \beta^2(\beta-1)^2\left(\frac{\sin(\pi\beta)}{\pi}\right)^{2(N-1)} \\ &\quad \prod_{j=2}^{N}\Gamma^2(1-\beta+j)\prod_{j=1}^{N-1}\Gamma^2(1+\beta+j)\ .\end{aligned} \tag{3.131}$$

Combining Eqs. (3.130) and (3.131), the squared modulus of the overlap acquires the form

$$\begin{aligned}|\langle\chi_0(\mathbf{n},n_y)|\Psi(\mathbf{m},J)\rangle|^2 &= \frac{\delta_{K,q_y}}{N+1}\left(\frac{\sin(\pi\beta)}{\pi\beta(\beta-1)}\right)^2\left(\frac{\Gamma(2-\beta)\Gamma(N+1)}{\Gamma(N+1-\beta)}\right)^2 \\ &\quad \left(\prod_{j=1}^{N-1}\frac{\Gamma^2(j+1)}{\Gamma(1-\beta+j)\Gamma(1+\beta+j)}\right)^2\ .\end{aligned} \tag{3.132}$$

In the large $N$ limit the ratio of Γ-functions in the first line can be expanded using Stirling's formula

$$\left(\frac{\Gamma(N+1)}{\Gamma(N+1-\beta)}\right)^2\approx\frac{1}{N^{2\beta}}\left(1+\frac{\beta(\beta-1)}{N}+\mathcal{O}(N^{-2})\right)\ . \tag{3.133}$$

To expand the product of Γ-functions in the second line of Eq. (3.132), we express it in terms of Barnes G-function $B_g(z)$ [91] according to

$$\prod_{j=1}^{N-1}\frac{\Gamma^2(j+1)}{\Gamma(1-\beta+j)\Gamma(1+\beta+j)}=\frac{B_g(2-\beta)B_g(2+\beta)B_g^2(N+1)}{B_g(N+1+\beta)B_g(N+1-\beta)}\ . \tag{3.134}$$

We use the logarithmic expansion of $B_g(z)$

$$\ln(B_g(z))=\frac{1}{12}-\ln(A)+\frac{z}{2}\ln(2\pi)+\left(\frac{z^2}{2}-\frac{1}{12}\right)-\frac{3z^2}{4}+\mathcal{O}(z^{-2})\ , \tag{3.135}$$

where $A\approx 1.2824$ is Glaisher's constant. The expansion above is valid for $z\in\mathbb{C}/\mathbb{R}^-$ with $|z|\gg 1$. In the large $N$ limit it yields for Eq. (3.134)

$$\prod_{j=1}^{N-1}\frac{\Gamma^2(j+1)}{\Gamma(1-\beta+j)\Gamma(1+\beta+j)}\approx\frac{B_g(2-\beta)B_g(2+\beta)e^{-\beta^2/2}}{N^{\beta^2}} \quad \text{for} \quad N\to\infty\ . \tag{3.136}$$

Combining Eqs. (3.133) and (3.136), we obtain the scaling for the modulus of the overlap integral in the limit $N \to \infty$

$$|\langle \chi_0(\mathbf{n}, n_y)|\Psi(\mathbf{m}, J)\rangle| \quad \propto \quad N^{-\beta(\beta+1)-1/2} \; . \tag{3.137}$$

Thus, for an arbitrary value of $\beta \in [0, 1]$, the overlap vanishes as $N \to \infty$. In particular for the ground state, where $\beta = 1/2$, the overlap vanishes like $N^{-5/4}$. This might be interpreted as an indicator for a non Fermi-liquid behavior if $|q_y| \leq k_{\mathrm{F}}$.

# Chapter 4

# Results for two spin-up Fermions

We study two minority Fermions that interact with a Fermi-sea of majority particles through repulsive contact interaction. As in the forgoing chapter, the reformulation of the eigenfunctions allows to calculate several quantities exactly. The outline of the chapter is as follows:

In Sec. 4.1, we specify the needed results from Chapter 2 to the present case. The system's total energy and an expression for the effective interaction energy between the minority particles are studied in Sec. 4.2. The two and three particle density-density correlation functions are discussed in Sec. 4.3. We summarize and conclude in Sec. 4.4.

## 4.1 Eigenfunctions and Bethe-Ansatz equations

If two spin-up (minority) particles are present the Hamiltonian of the system reads

$$H = -\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2} - \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 4c \sum_{j=1}^{N} \left(\delta(x_j - y_1) + \delta(x_j - y_2)\right) \ . \tag{4.1}$$

For the exact eigenfunctions we have according to Theorem 1

$$\Psi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda}) \propto \sum_{R \in S(2)} \mathrm{sgn}(R) \left[\imath(\Lambda_{R1} - \Lambda_{R2}) + 2c\mathrm{sgn}(y_2 - y_1)\right] \Phi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda}) \ , \tag{4.2}$$

where

$$\Phi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda}) = \det \Big[ A_j(\Lambda_{R1}, x_l - y_1) A_j(\Lambda_{R2}, x_l - y_2) e^{\imath k_j x_l} \Big| \Big| A_j(\Lambda_{R2}, y_1 - y_2) e^{\imath k_j y_1} \Big| A_j(\Lambda_{R1}, y_2 - y_1) e^{\imath k_j y_2} \Big]_{\substack{j=1,\ldots,N+2 \\ l=1,\ldots,N}} \tag{4.3}$$

and we recall that

$$A_j(\Lambda, x) = \imath(k_j - \Lambda) + c\mathrm{sgn}(x) \ . \tag{4.4}$$

From the Bethe-Ansatz equations (2.32) and (2.33) we obtain the quantization rules for the present case

$$k_j L = 2\pi n_j - 2\arctan\left(\frac{k_j-\Lambda_1}{c}\right) + \arctan\left(\frac{k_j-\Lambda_2}{c}\right) \qquad j=1,\ldots,N+2 \quad , \tag{4.5}$$

$$2\pi J_\mu = 2\sum_{j=1}^{N+2}\arctan\left(\frac{k_j-\Lambda_\mu}{c}\right) + 2\arctan\left(\frac{\Lambda_\mu-\Lambda_\nu}{2c}\right) \qquad \mu\neq\nu\in\{1,2\}\ . \tag{4.6}$$

The quantum numbers $J_\mu$ and $n_j$ are integers and $-(N+3)/2 \leq J_\mu \leq (N+3)/2$. For the groundstate the quantum numbers $n_j$ are

$$\{n_j\}_{j=1,\ldots,N+2} = \left\{-\frac{N+1}{2},\ldots,0,\ldots,\frac{N+1}{2}\right\}\ . \tag{4.7}$$

As in the forgoing chapter, we assume the majority Fermions to be in the ground state such that their non-interacting momentum distribution is given by a Fermi-sea. Upon this we consider excitations where the two spin-up particles can occupy arbitrary states within the Fermi-sea. In complete analogy to the case of a single spin-up particle, the quantum numbers $J_1$ and $J_2$ indicate the single particle states which for vanishing interaction are doubly occupied. If $J_{1,2} = \pm(N+1)/2$, the spin-up particles boarder the Fermi-sea, whereas for $J_{1,2} = 0$ the corresponding quasi-momenta is in the core of the Fermi-sea. The overall momentum is now $K = -2\pi(J_1+J_2)/L \equiv K_1 + K_2$, where we recall the definition $K_\mu = -2\pi J_\mu/L$. In the thermodynamic limit $K_\mu$ takes its values in between $\pm k_{\mathrm{F}}$ as $J_\mu$ varies in between the bounds given above. In the lab frame, where the Fermi-sea is at rest, $K_1$ and $K_2$ can be identified with the free momenta of the minority particles.

## 4.2 Energy and effective interaction energy

In Sec. 4.2.1, the energy shift due to the interaction is calculated. Based on that, we study in Sec. 4.2.2 sub-leading terms of the energy shift which allow for an interpretation as effective interaction energy.

### 4.2.1 Energy

We consider the energy for the case that $M$ spin-up Fermions are present. Employing the density of states (2.49), the system's total energy can be expressed as

$$E \;=\; \int_{k_-}^{k_+} dk k^2 \varrho(k) \;=\; \frac{L(k_+^3-k_-^3)}{6\pi} + \frac{1}{\pi}\sum_{\mu=1}^{M}\int_{k_-}^{k_+} dk \frac{ck^2}{(k-\Lambda_\mu)^2+c^2}\ , \tag{4.8}$$

where $k_\pm$ is defined via Eq. (2.51). The equation above can be expanded in inverse powers of the system size. We start with the first term on the right hand side. Taking

into account the corrections of $k_\pm$ up to order $L^{-1}$ we have

$$\begin{aligned}\frac{L(k_+^3-k_-^3)}{6\pi} = E_{\rm F} &+ k_{\rm F}^2\sum_{\mu=1}^{M}\left(1+\frac{\partial u(\Lambda_\mu)}{\partial k_{\rm F}}\right) \\ &+ \frac{\pi k_{\rm F}}{2L}\sum_{\sigma=\pm1}\sum_{\mu,\nu=1}^{M}\left[1-\frac{1}{\pi}\arctan\left(\frac{k_{\rm F}+\sigma\Lambda_\mu}{c}\right)\right]^2 .\end{aligned} \tag{4.9}$$

We recall that $u(\Lambda)$ is defined through Eq. (2.54). Note that the terms in the first line of the equation above are additive in $\Lambda_\mu$, whereas the terms in the second line couple $\Lambda_\mu$ and $\Lambda_\nu$.

We consider the integral on the right hand side of Eq. (4.8). Terms of order $L^{-1}$ enter here via the boundaries $k_\pm$. Expanding this term we obtain

$$\begin{aligned}\frac{1}{\pi}\sum_{\mu=1}^{M}\int_{-k_-}^{+k_+}dk\frac{ck^2}{(k-\Lambda_\mu)^2+c^2} &= \sum_{\mu=1}^{M}\frac{1}{\pi}\int_{-k_{\rm F}}^{+k_{\rm F}}dk\frac{ck^2}{(k-\Lambda_\mu)^2+c^2} \\ &+ \sum_{\sigma=\pm1}\sum_{\mu,\nu=1}^{M}\frac{ck_{\rm F}^2}{(k_{\rm F}+\sigma\Lambda_\mu)^2+c^2}\left[1-\frac{1}{\pi}\arctan\left(\frac{k_{\rm F}+\sigma\Lambda_\mu}{c}\right)\right] ,\end{aligned} \tag{4.10}$$

where the terms in the first line are again additive in $\Lambda_\mu$, whereas those in the second line mix $\Lambda_\mu$ and $\Lambda_\nu$. Combining Eqs. (4.9) and (4.10), the total energy now might be written as

$$E = E_{\rm F} + \sum_{\mu=1}^{M}E^{(1)}(\Lambda_\mu) + \sum_{\mu<\nu}^{M}E^{(2)}(\Lambda_\mu,\Lambda_\nu) + \mathcal{O}(L^{-2}) . \tag{4.11}$$

The leading order term is the Fermi-energy $E_{\rm F} = Lk_{\rm F}^3/(3\pi)$. The single particle energy $E^{(1)}(\Lambda_\mu)$ is defined as in Eq. (3.18) and is of the order one. The corrections up to order $L^{-1}$ to that expression can be extracted from Eqs. (4.9) and (4.10). In marked contrast, the leading order term of the two particle energy $E^{(2)}(\Lambda_\mu,\Lambda_\nu)$ is of order $L^{-1}$. From the equations above it can be identified as

$$\begin{aligned}E^{(2)}(\Lambda_\mu,\Lambda_\nu) &= \sum_{\sigma=\pm1}\frac{2k_{\rm F}}{\pi L}\left(2-k_{\rm F}\frac{d}{dk_{\rm F}}\right) \\ &\left[\frac{\pi}{2}-\arctan\left(\frac{k_{\rm F}+\sigma\Lambda_\mu}{c}\right)\right]\left[\frac{\pi}{2}-\arctan\left(\frac{k_{\rm F}+\sigma\Lambda_\nu}{c}\right)\right]+\mathcal{O}(L^{-2}) .\end{aligned} \tag{4.12}$$

Since the energy of the non-interacting system is

$$E_0 = E_{\rm F} + \sum_{\mu=1}^{M}K_\mu^2 , \tag{4.13}$$

the total energy shift $\Delta E_M = E - E_0$ for $M$ spin-up particles can in leading order of the system size be expressed as

$$\Delta E_M(\mathbf{K}) = \sum_{\mu=1}^{M} \left( E^{(1)}(\Lambda_\mu) - K_\mu^2 \right) + \mathcal{O}(L^{-1}) \,, \tag{4.14}$$

where $\Lambda_\mu$ is according to Eq. (2.53) determined by $K_\mu$ only. Thus, the energy shift is in leading order additive i.e. it is the sum of $M$ *independent* terms, where each contribution corresponds to the energy shift caused by the interaction of a single spin-up particle with the Fermi-sea. One might argue that this is expected, since there is no interaction between the spin-up particles. The results for one spin-up particle hence directly carry over to the present case. In particular the energy shift vanishes for $c \to 0$ and saturates at a value $E_{\max} = \sum_\mu^M (k_\mathrm{F} - K_\mu^2)$ for infinite strong interaction strength. In between these two values the energy shift increases monotonically. However, we emphasize that in sub-leading order $\Lambda_\mu = \Lambda_\mu(\mathbf{K})$ depends on all initial momenta $K_\mu$, $\mu = 1, \ldots, M$ and thus the lower order terms are not additive.

### 4.2.2 Effective interaction energy for two spin-up Fermions

The single particle energy $E^{(1)}(\Lambda_\mu)$ is additive as function of $\Lambda_\mu$, however it is not additive as function of the free momenta $K_\mu$. Thus the functional form of the total energy shift changes when a spin-up particle is added to the system. The total energy shift of a system of $M$ spin-up particles $\Delta E_M$ can be expanded in a cluster expansion as

$$\begin{aligned}
\Delta E_1(K_1) &= E^{(1)}(\Lambda_1)\Big|_{\Lambda_1 = u^{-1}(K_1)} - K_1^2 \\
\Delta E_2(K_1, K_2) &= \Delta E_1(K_1) + \Delta E_1(K_2) + W_2(K_1, K_2) \\
\Delta E_3(K_1, K_2, K_3) &= \sum_{\mu=1}^{3} \Delta E_1(K_\mu) + \sum_{\mu>\nu}^{3} W_2(K_\mu, K_\nu) + W_3(K_1, K_2, K_3) \\
&\vdots
\end{aligned} \tag{4.15}$$

The first term in this expansion contains the information about the single particle dispersion and has been discussed in Sec. 3.2.1 of the forgoing Chapter. Here we discuss the second term, which contains the information about the interaction energy between the two spin-up particles in the presence of the Fermi-sea.

The effective interaction energy $W_2(K_1, K_2)$ is a function of $K_1$ and $K_2$ rather than of $\Lambda_1$ and $\Lambda_2$. Therefore it is not just given by the two-particle energy $E^{(2)}(\Lambda_1, \Lambda_2)$ but also the single particle energies $E^{(1)}(\Lambda_\mu)$ contribute. Using Eqs. (2.53) and (2.54) one

finds

$$W_2(K_1,K_2) = -\frac{\partial E^{(1)}(\Lambda_1)}{\partial \Lambda_1}\left(\frac{\partial u(\Lambda_1,k_\mathrm{F})}{\partial \Lambda_1}\right)^{-1} w_2(\Lambda_1,\Lambda_2) - \frac{\partial E^{(1)}(\Lambda_2)}{\partial \Lambda_2}\left(\frac{\partial u(\Lambda_2,k_\mathrm{F})}{\partial \Lambda_2}\right)^{-1} w_2(\Lambda_2,\Lambda_1) + E^{(2)}(\Lambda_1,\Lambda_2)\Big|_{\substack{\Lambda_1=u^{-1}(K_1)\\ \Lambda_2=u^{-1}(K_2)}}, \tag{4.16}$$

which has for general $c > 0$ to be treated numerically, since the function $u(\Lambda)$ in Eq. (2.54) can not be inverted analytically. Nevertheless an expansion of Eq. (4.16) for small interaction strength is possible and yields for $|K_{1,2}| < k_\mathrm{F}$

$$\lim_{c\to 0^+} W_2(K_1,K_2) = -\frac{4cK_1K_2}{L}\left(\frac{1}{k_\mathrm{F}^2-K_1^2}+\frac{1}{k_\mathrm{F}^2-K_2^2}\right). \tag{4.17}$$

This expression becomes singular if either of the two momenta approaches the Fermi-momentum. An asymptotic expansion of Eq. (4.17) for strong interaction yields

$$W_2(K_1,K_2) = \frac{2\pi k_\mathrm{F}}{L}\left\{1+\frac{K_1K_2}{k_\mathrm{F}^2}-\frac{2k_\mathrm{F}}{c\pi}\left[\cos^2\left(\frac{\pi K_1}{2k_\mathrm{F}}\right)+\cos^2\left(\frac{\pi K_2}{2k_\mathrm{F}}\right)\right]\right\}+\mathcal{O}(c^{-2}). \tag{4.18}$$

In Figs. 4.1 and 4.2, the function $W_2(K_1,K_2)L/k_\mathrm{F}$ is plotted for the choices $K_1 = -K_2$ and $K_1 = K_2$ of momenta, respectively. It is seen that the effective interaction energy has a maximum for very small coupling strength. For strong coupling strength it decays to the value given by equation (4.18). Only for $K_1 = -K_2 = k_\mathrm{F}$ it decays to zero. The effective interaction energy depends non-trivially on both free momenta $K_1$ and $K_2$. This implies that translation invariance of the reduced system, consisting only of the two spin-up Fermions, is broken. The dependence on the total momentum becomes most striking in the weak coupling limit. As the expansion (4.17) shows, $W_2(K_1,K_2)$ is positive for $K_1K_2 < 0$. However, if $K_1K_2 > 0$, Eq. (4.17) reveals that $W(K_1,K_2)$ becomes negative for small values of $c$.

## 4.3 Density-density correlation functions

Using the determinantal representation (4.2) of the many-body wave function, the density-density correlation function of the two spin-up Fermions and even the three point correlation function of the two spin-up and one spin-down particle can be calculated exactly. The latter yields insight to what extent the two impurities affect the otherwise flat density profile of the Fermi-sea.

In general the equal time density-density correlation function for the two spin-up and $n$-spin-down Fermions is defined in coordinate representation trough the multiple integral

$$R_n(y_1,y_2,x_1,\ldots,x_n) := 4N^n \int_0^L dx_{n+1}\cdots\int_0^L dx_N \left|\Psi(\mathbf{x},\mathbf{k},y_1,y_2,\Lambda_1,\Lambda_2)\right|^2. \tag{4.19}$$

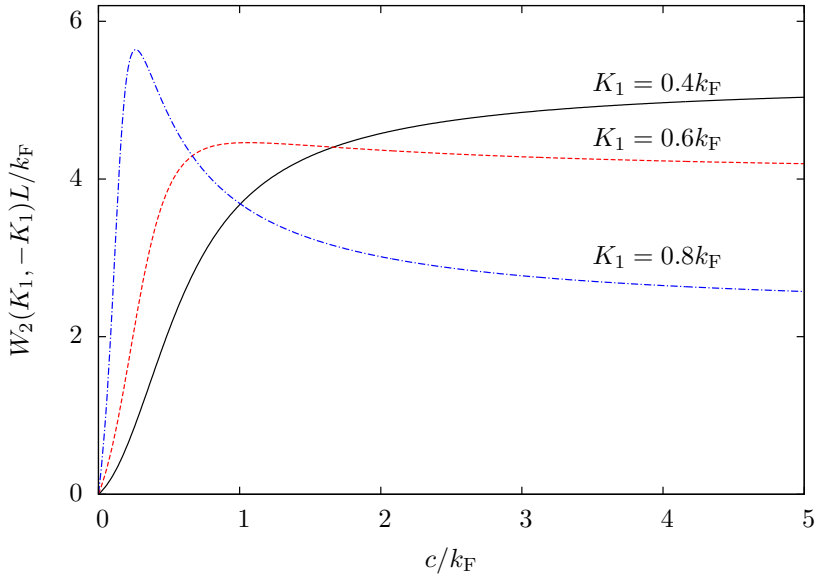


Figure 4.1: Interaction energy $W_2(K_1, K_2)$ for $K_1 = -K_2$ and different values of the free momenta $K_1$. The values are: $K_1 = 0.4k_\mathrm{F}$ (solid line, black), $K_1 = 0.6k_\mathrm{F}$ (dashed line, red), $K_1 = 0.8k_\mathrm{F}$ (dot-dashed line, blue) .

According to its definition, we have $R(y_1, y_2, x_1, \ldots, x_n) = R(y_2, y_1, x_1, \ldots, x_n)$ and thus the discussion can be restricted to the case $y_1 < y_2$. Using the explicit form of the eigenfunctions (4.2) for $M = 2$, this can be written as

$$R_n(y_1, y_2, x_1, \ldots, x_n) \propto \sum_{\substack{R,R' \\ \in S(2)}} \mathrm{sgn}(R + R') \left[\imath(\Lambda_{R1} - \Lambda_{R2}) + 2c\right] \left[\imath(\Lambda_{R'2} - \Lambda_{R'1}) - 2c\right] \quad (4.20)$$
$$\times \int_0^L dx_{n+1} \cdots \int_0^L dx_N \Phi(\mathbf{x}, y_1, y_2, \Lambda_{R1}, \Lambda_{R2}) \Phi^*(\mathbf{x}, y_1, y_2, \Lambda_{R'1}, \Lambda_{R'2}) \; .$$

Note that the quantities $\Lambda_{1,2}$ and therefore the correlation function itself depends on the momenta $K_1$, $K_2$ of the free spin-up particles. In particular we aim at evaluating Eq. (4.20) for the cases $n = 0$ and $n = 1$. While in the first case Eq. (4.20) corresponds to the density-density correlation function of the two spin-up Fermions it yields for $n = 1$ the three particle density-density correlation function of the two spin-up Fermions and a spin-down particle. The evaluation of Eq. (4.20) is straight forward but tedious. The details of the derivations are presented in Secs. A.5 and A.6 of Appendix A. As shown there, the outcome for Eq. (4.20) for $n = 0, 1$ can be cast into the unified form

$$\hat{R}_n(\hat{y}_1, \hat{y}_2, \hat{x}_n) = \frac{\mathcal{R}}{2^n} \left\{ \det[I^{(n)}(\hat{\Lambda}_1, \hat{\Lambda}_2)] - \mathrm{Re}\left( e^{2\imath \arctan\left(\frac{\Lambda_1 - \Lambda_2}{2c}\right)} \det[J^{(n)}(\hat{\Lambda}_1, \hat{\Lambda}_2)] \right) \right\} \quad (4.21)$$

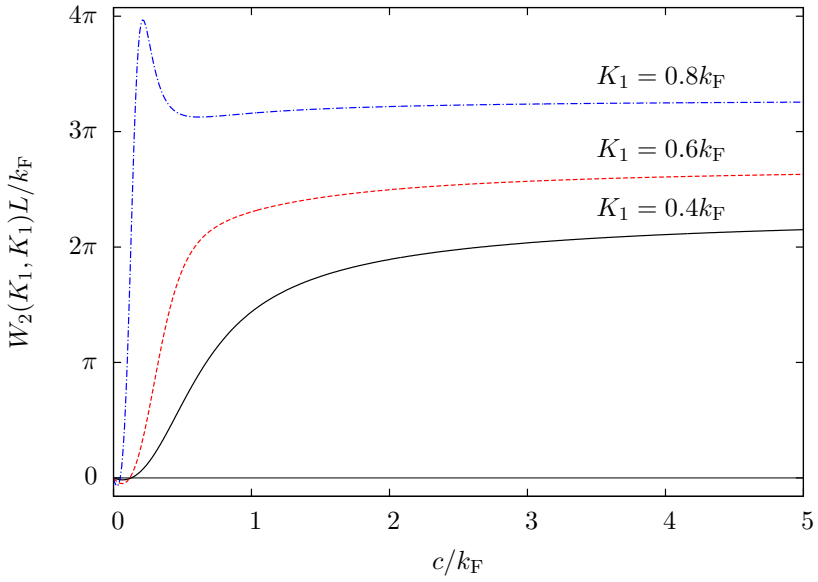


Figure 4.2: Interaction energy $W_2(K_1, K_2)$ for $K_1 = K_2$ and different values of the free momentum $K_1$. The values are: $K_1 = 0.4k_F$ (solid line, black), $K_1 = 0.6k_F$ (dashed line, red), $K_1 = 0.8k_F$ (dot-dashed line, blue) .

where the normalization constant $\mathcal{R}$ reads

$$\mathcal{R}^{-1} = \prod_{j=1}^{2} \int_{-1}^{+1} dk \frac{1}{(k-\hat{\Lambda}_j)^2 + \hat{c}^2} - \mathrm{Re}\left( \int_{-1}^{+1} dk \frac{e^{i \arctan\left(\frac{\hat{\Lambda}_1 - \hat{\Lambda}_2}{2\hat{c}}\right)}}{[-\imath(k-\hat{\Lambda}_1) + \hat{c}][\imath(k-\hat{\Lambda}_2) + \hat{c}]} \right)^2 \tag{4.22}$$

and the dimensionless quantities

$$\begin{aligned} &\hat{y}_{1,2} = y_{1,2}\ k_F \quad , \quad \hat{x}_1 = x k_F \quad , \quad \hat{\Lambda}_{1,2} = \frac{\Lambda_{1,2}}{k_F} \quad , \quad \hat{c} = \frac{c}{k_F} \\ &\hat{R}_n(\hat{y}_1, \hat{y}_2, \hat{x}_n) = \frac{L^{2+n}}{4N^n} R_n(y_1, y_2, x_n) \end{aligned} \tag{4.23}$$

have been introduced. In Eq. (4.21), $I^{(n)}(\hat{\Lambda}_1, \hat{\Lambda}_2) = [I_{jl}(\hat{\Lambda}_1, \hat{\Lambda}_2)]_{j,l=1,\ldots,n+2}$ denotes a $(n+2) \times (n+2)$ matrix. Furthermore $J^{(n)}(\hat{\Lambda}_1, \hat{\Lambda}_2) = [J_{jl}(\hat{\Lambda}_1, \hat{\Lambda}_2)]_{j,l=1,\ldots,n+2}$ is a matrix of the same dimension. For both cases, $n = 0, 1$, the entries are one-fold integrals. Their explicit form will be stated below. Beside its coordinate dependence Eq. (4.21) depends on the interaction strength $c$ and $\Lambda_1$ and $\Lambda_2$. The latter two quantities are determined by Eq. (2.53). In the following we treat the two cases $n = 0, 1$ separately.

**Two particle density-density correlation function**

To complete the description of Eq. (4.21) for $n = 0$ we need to give the explicit form of the entries of the matrices $I^{(0)}(\hat{\Lambda}_1, \hat{\Lambda}_2)$ and $J^{(0)}(\hat{\Lambda}_1, \hat{\Lambda}_2)$. From the derivation presented in the Sec. A.5 it follows that these are

$$\begin{aligned}
I^{(0)}_{jj}(\hat{\Lambda}_1, \hat{\Lambda}_2) &= \int_{-1}^{+1} dk \frac{1}{(k-\hat{\Lambda}_j)^2 + \hat{c}^2} \ , \qquad j = 1, 2 \ , \\
I^{(0)}_{12}(\hat{\Lambda}_1, \hat{\Lambda}_2) &= \int_{-1}^{+1} dk \frac{e^{\imath k(\hat{y}_1 - \hat{y}_2)}}{(\imath(k-\hat{\Lambda}_1) + \hat{c})(\imath(k-\hat{\Lambda}_2) + \hat{c})} \ , \\
J^{(0)}_{jj}(\hat{\Lambda}_1, \hat{\Lambda}_2) &= \int_{-1}^{+1} dk \frac{1}{(-\imath(k-\hat{\Lambda}_1) + \hat{c})(\imath(k-\hat{\Lambda}_2) + \hat{c})} \ , \qquad j = 1, 2 \ , \\
J^{(0)}_{12}(\hat{\Lambda}_1, \hat{\Lambda}_2) &= \int_{-1}^{+1} dk \frac{e^{\imath k(\hat{y}_1 - \hat{y}_2) - 2i \arctan((k-\Lambda_2)/c)}}{(k-\hat{\Lambda}_1)^2 + \hat{c}^2} \ .
\end{aligned} \tag{4.24}$$

The entries for $j > l$ are obtained from those for $j < l$ by exchanging $\hat{\Lambda}_1 \leftrightarrow \hat{\Lambda}_2$ and taking the complex conjugated i.e.

$$I^{(0)}_{jl}(\hat{\Lambda}_1, \hat{\Lambda}_2) = \left[ I^{(0)}_{lj}(\hat{\Lambda}_2, \hat{\Lambda}_1) \right]^* \quad , \quad J^{(0)}_{jl}(\hat{\Lambda}_1, \hat{\Lambda}_2) = \left[ J^{(0)}_{lj}(\hat{\Lambda}_2, \hat{\Lambda}_1) \right]^* \ . \tag{4.25}$$

The integrals in Eq. (4.24) reveal that $\hat{R}_0(\hat{y}_1, \hat{y}_2)$ is a function of the difference $\hat{y}_1 - \hat{y}_2$ only, as expected by translation invariance.

We consider the limiting behavior of $\hat{R}_0(\hat{y}_1, \hat{y}_2)$ for infinitely strong and vanishing interaction strength. For $\hat{c} \to 0^+$ we use the representation of the $\delta$-function as Lorentzian distribution with vanishing width

$$\lim_{c \to 0^+} \frac{1}{\pi} \frac{c}{(k-\Lambda)^2 + c^2} = \delta(k - \Lambda) \ . \tag{4.26}$$

This allows to evaluate the integrals in Eqs. (4.22) and (4.24). Doing so, one finds in the limit of vanishing interaction strength the density-density correlation function of two free Fermions with momenta $\hat{K}_1 = \hat{\Lambda}_1$ and $\hat{K}_2 = \hat{\Lambda}_2$, that is

$$\lim_{\hat{c} \to 0^+} \hat{R}_0(\hat{y}_1, \hat{y}_2) = 1 - \cos\left( (\hat{K}_1 - \hat{K}_2)(\hat{y}_1 - \hat{y}_2) \right) \ . \tag{4.27}$$

For hardcore interaction, the dependence on the integration variable $k$ in Eqs. (4.22) and (4.24) drop out and the integrals become trivial. Then it is easily shown that

$$\lim_{\hat{c} \to +\infty} \hat{R}_0(\hat{y}_1, \hat{y}_2) = 1 - \left( \frac{\sin(\hat{y}_1 - \hat{y}_2)}{\hat{y}_1 - \hat{y}_2} \right)^2 \ . \tag{4.28}$$

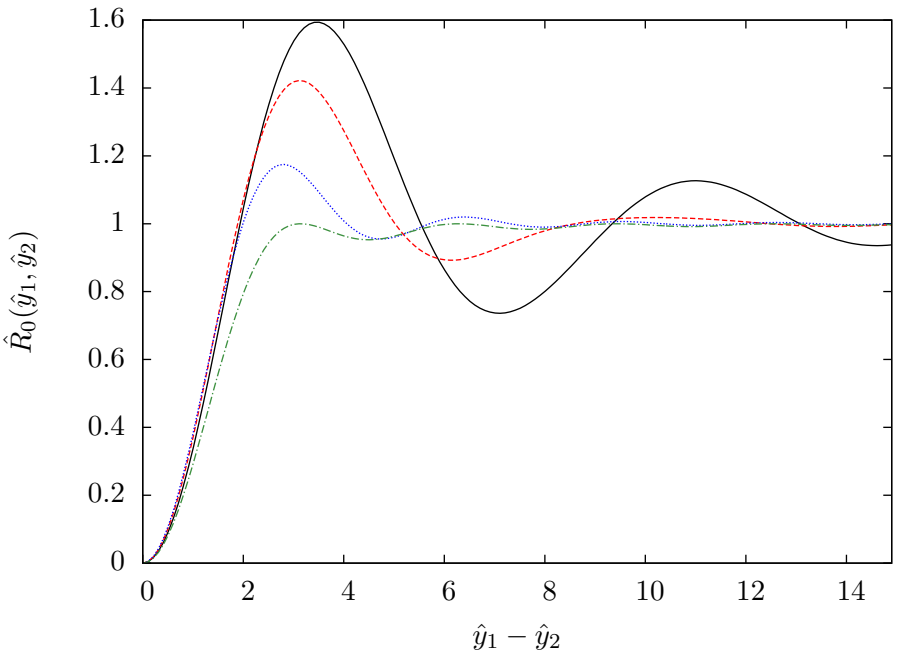


Figure 4.3: Two particle density-density correlation function as function of the distance $\hat{y}_1 - \hat{y}_2$ for the quantum numbers $\hat{K}_1 = -\hat{K}_2 = \pi/8$. The values for the interaction strength $\hat{c}$ are: $\hat{c} = 0.05$ (solid line, black), $\hat{c} = 0.1$ (dashed line, red), $\hat{c} = 0.7$ (dotted line, blue) and $\hat{c} = +\infty$ (dot-dashed line, green).

This corresponds to the density-density correlation function of two free Fermions that form part of the Fermi–sea. In Fig. 4.3 we show the plot of $\hat{R}_0(\hat{y}_1, \hat{y}_2)$ for the choice $\hat{K}_1 = -\hat{K}_2 = \pi/8$ and for different values of $\hat{c}$. As $\hat{c}$ varies from zero to infinity the two particle density-density correlation function undergoes a transition from that for two free Fermions to the one of a non-interacting Fermi-sea. For other choices of $\hat{K}_1$ and $\hat{K}_2$ the picture remains qualitatively similar.

### Three particle density-density correlation function

In contrast to the two particle case, the matrices $I^{(1)}(\hat{\Lambda}_1, \hat{\Lambda}_2)$ and $J^{(1)}(\hat{\Lambda}_1, \hat{\Lambda}_2)$ have now dimension $3 \times 3$. The entries for $j, l = 1, 2$ are identical with those for the two particle density-density correlation function given in Eq. (4.24) i.e.

$$I^{(1)}_{jl}(\hat{\Lambda}_1, \hat{\Lambda}_2) = I^{(0)}_{jl}(\hat{\Lambda}_1, \hat{\Lambda}_2) \quad , \quad J^{(1)}_{jl}(\hat{\Lambda}_1, \hat{\Lambda}_2) = J^{(0)}_{jl}(\hat{\Lambda}_1, \hat{\Lambda}_2) \quad \text{for} \quad j, l = 1, 2 \; . \quad (4.29)$$

Thus it remains to give the entries of the third column and the third row of the matrices $I^{(1)}(\hat{\Lambda}_1, \hat{\Lambda}_2)$ and $J^{(1)}(\hat{\Lambda}_1, \hat{\Lambda}_2)$. As shown in the Sec. A.6 of Appendix A these are

$$I^{(1)}_{13}(\hat{\Lambda}_1, \hat{\Lambda}_2) = \int_{-1}^{+1} dk \frac{[-\imath(k-\hat{\Lambda}_1) + \hat{c}\mathrm{sgn}(\hat{x}-\hat{y}_1)][-\imath(k-\hat{\Lambda}_2) + \hat{c}\mathrm{sgn}(\hat{x}-\hat{y}_2)]}{[(k-\hat{\Lambda}_1)^2+\hat{c}^2][\imath(k-\hat{\Lambda}_2)+\hat{c}]} e^{\imath k(\hat{y}_1-\hat{x})} ,$$

$$I^{(1)}_{23}(\hat{\Lambda}_1, \hat{\Lambda}_2) = \int_{-1}^{+1} dk \frac{[-\imath(k-\hat{\Lambda}_1) + \hat{c}\mathrm{sgn}(\hat{x}-\hat{y}_1)][-\imath(k-\hat{\Lambda}_2) + \hat{c}\mathrm{sgn}(\hat{x}-\hat{y}_2)]}{[(k-\hat{\Lambda}_2)^2+\hat{c}^2][-\imath(k-\hat{\Lambda}_1)+\hat{c}]} e^{\imath k(\hat{y}_2-\hat{x})} ,$$

$$J^{(1)}_{13}(\hat{\Lambda}_1, \hat{\Lambda}_2) = \int_{-1}^{+1} dk \frac{[-\imath(k-\hat{\Lambda}_2) + \hat{c}\mathrm{sgn}(\hat{x}-\hat{y}_1)][-\imath(k-\hat{\Lambda}_1) + \hat{c}\mathrm{sgn}(\hat{x}-\hat{y}_2)]}{[(k-\hat{\Lambda}_1)^2+\hat{c}^2][\imath(k-\hat{\Lambda}_2)+\hat{c}]} e^{\imath k(\hat{y}_1-\hat{x})} ,$$

$$J^{(1)}_{23}(\hat{\Lambda}_1, \hat{\Lambda}_2) = \int_{-1}^{+1} dk \frac{[-\imath(k-\hat{\Lambda}_1) + \hat{c}\mathrm{sgn}(\hat{x}-\hat{y}_1)][-\imath(k-\hat{\Lambda}_2) + \hat{c}\mathrm{sgn}(\hat{x}-\hat{y}_2)]}{[(k-\hat{\Lambda}_2)^2+\hat{c}^2][-\imath(k-\hat{\Lambda}_1)+\hat{c}]} e^{\imath k(\hat{y}_2-\hat{x})} ,$$

$$J^{(1)}_{33}(\hat{\Lambda}_1, \hat{\Lambda}_2) = \int_{-1}^{+1} dk [-\imath(k-\hat{\Lambda}_{R1}) + \hat{c}\mathrm{sgn}(\hat{x}-\hat{y}_1)][\imath(k-\hat{\Lambda}_{R'1}) + \hat{c}\mathrm{sgn}(\hat{x}-\hat{y}_1)]$$
$$\frac{[-\imath(k-\hat{\Lambda}_{R2}) + \hat{c}\mathrm{sgn}(\hat{x}-\hat{y}_2)][\imath(k-\hat{\Lambda}_{R'2}) + \hat{c}\mathrm{sgn}(\hat{x}-\hat{y}_2)]}{[(k-\hat{\Lambda}_1)^2+\hat{c}^2][(k-\hat{\Lambda}_2)^2+\hat{c}^2]} . \qquad (4.30)$$

Furthermore $I^{(1)}_{33}(\hat{\Lambda}_1, \hat{\Lambda}_2) = 2$. Again the entries for $j < l$ are obtained from those for $l > j$ by exchanging $\Lambda_1$ and $\Lambda_2$ and taking the complex conjugated i.e. $I^{(1)}_{3n}(\hat{\Lambda}_1, \hat{\Lambda}_2) = [I^{(1)}_{n3}(\hat{\Lambda}_1, \hat{\Lambda}_2)]^*$, $J^{(1)}_{3n}(\hat{\Lambda}_1, \hat{\Lambda}_2) = [J^{(1)}_{n3}(\hat{\Lambda}_2, \hat{\Lambda}_1)]^*$ for $n = 1, 2$.

We consider the limiting behavior of $\hat{R}_1(\hat{y}_1, \hat{y}_2, \hat{x})$ for infinitely strong and vanishing interaction strength. For $\hat{c} \to 0^+$ we make use of relation (4.26) to evaluate the normalization constant (4.22) and the integrals in Eq. (4.30). As before the outcome is given by the two particle density-density correlation function of two free Fermions with momenta $\hat{K}_1 = \hat{\Lambda}_1$ and $\hat{K}_2 = \hat{\Lambda}_2$ such that

$$\lim_{\hat{c}\to 0^+} \hat{R}_1(\hat{y_1}, \hat{y_2}, \hat{x}) = 1 - \cos\Big((\hat{K}_1 - \hat{K}_2)(\hat{y}_1 - \hat{y}_2)\Big) . \qquad (4.31)$$

To evaluate $\hat{R}_1(\hat{y}_1, \hat{y}_2, \hat{x})$ for infinitely strong interaction we recall the definition of the quantities $\lambda_{1,2} = \lim_{c\to\infty} \hat{\Lambda}_{1,2}/\hat{c} = -\tan(\pi K_{1,2}/2k_\mathrm{F})$. They vary from $-\infty$ to $+\infty$ as $K_{1,2}$ vary from $k_\mathrm{F}$ to -$k_\mathrm{F}$. In the hardcore limit the integrals in Eqs. (4.30) become elementary and can easily be carried out. Then after some algebra the resulting expression for $\hat{R}_1(\hat{y_1}, \hat{y_2}, \hat{x})$ can be cast into the form

$$\lim_{c\to+\infty} \hat{R}_1(\hat{y}_1, \hat{y}_2, \hat{x}) = \mathcal{W}(\lambda_1, \lambda_2)\hat{R}_{FF}(\hat{y_1}, \hat{y_2}, \hat{x}) , \qquad (4.32)$$

where we have introduced the quantity

$$\hat{R}_{FF}(\hat{y}_1, \hat{y}_2, \hat{x}) = 1 \quad - \quad \left(\frac{\sin(\hat{x}-\hat{y}_1)}{\hat{x}-\hat{y}_1}\right)^2 - \left(\frac{\sin(\hat{x}-\hat{y}_2)}{\hat{x}-\hat{y}_2}\right)^2 - \left(\frac{\sin(\hat{y}_1-\hat{y}_2)}{\hat{y}_1-\hat{y}_2}\right)^2 \qquad (4.33)$$

$$+ \quad 2\frac{\sin(\hat{x}-\hat{y}_1)}{\hat{x}-\hat{y}_1}\frac{\sin(\hat{x}-\hat{y}_2)}{\hat{x}-\hat{y}_2}\frac{\sin(\hat{y}_1-\hat{y}_2)}{\hat{y}_1-\hat{y}_2} \ ,$$

which corresponds to the three particle density-density correlation function of free Fermions. The factor $\mathcal{W}(\lambda_1,\lambda_2)$ in Eq. (4.32) is given by

$$\mathcal{W}(\lambda_1,\lambda_2) = \frac{1-\mathrm{Re}\,\mathcal{U}\mathcal{V}}{1-\mathrm{Re}\,\mathcal{U}} \tag{4.34}$$

with

$$\mathcal{U} = \frac{(1+\lambda_1^2)(1+\lambda_2^2)}{(\lambda_1-\lambda_2)^2+4}\left(\frac{\imath(\lambda_1-\lambda_2)+2}{(1+\imath\lambda_1)(1-\imath\lambda_2)}\right)^2 \ , \tag{4.35}$$

$$\mathcal{V} = \frac{(\imath\lambda_1+\mathrm{sgn}(\hat{x}-\hat{y}_2))(\imath\lambda_2+\mathrm{sgn}(\hat{x}-\hat{y}_1))}{(\imath\lambda_1+\mathrm{sgn}(\hat{x}-\hat{y}_1))(\imath\lambda_2+\mathrm{sgn}(\hat{x}-\hat{y}_2))} \ . \tag{4.36}$$

According to the equations above, $\mathcal{V}$ and hence also $\mathcal{W}(\lambda_1,\lambda_2)$ is a stepwise constant function of $\hat{x}$. If $\hat{x}$ lies outside the the interval $(\hat{y}_1,\hat{y}_2)$ we have $\mathcal{V}=1$ such that $\hat{R}_1(\hat{y}_1,\hat{y}_2,\hat{x}) = \hat{R}_{FF}(\hat{y}_1,\hat{y}_2,\hat{x})$ for $\hat{x}\notin(\hat{y}_1,\hat{y}_2)$ . However, if $\hat{x}$ lies in between the two spin-up particles, the function $\mathcal{W}(\lambda_1,\lambda_2)$ yields a weight for the density of the Fermi-sea inside the interval $(\hat{y}_1,\hat{y}_2)$, which crucially depends on the quantities $\lambda_1$ and $\lambda_2$. The choice $K_1=-K_2$ also implies $\lambda_1=-\lambda_2$ and Eq. (4.34) simplifies further

$$\mathcal{W}(\lambda_1,-\lambda_1) = \frac{(\lambda_1^2-3)^2}{(1+\lambda_1^2)^2} \ . \tag{4.37}$$

Thus if $K_1=\pm k_\mathrm{F}$ boarders the Fermi-sea such that $\lambda_1=\pm\infty$, we have $\mathcal{W}(\lambda_1,-\lambda_1)=1$ and consequently the three particle density-density correlation function coincides with that of free Fermions for all values of $\hat{x}$. Most interestingly Eq. (4.37) exhibits a root at $\lambda_1=\sqrt{3}$ which corresponds to the choice $K_1=-K_2=2k_\mathrm{F}/3$. This implies that the density of the Fermi-sea between $\hat{y}_1$ and $\hat{y}_2$ vanishes identically. On the other hand, if $K_2=0$ is chosen to be in the core of the Fermi-sea, $\mathcal{W}(\lambda_1,0)$ diverges as $K_1\to\pm k_\mathrm{F}$ approaches the border of the Fermi-sea. Consequently the density of the spin-down particles in the region $(\hat{y}_1,\hat{y}_2)$ increases as $K_1\to\pm k_\mathrm{F}$ and finally diverges for $K_1=\pm k_\mathrm{F}$.

For finite $\hat{c}$ the integrals (4.30) seem not to allow for a further analytical evaluation. However, numerically our result can easily be implemented. For fixed $\hat{y}_1$ and $\hat{y}_2$ the three particle density-density correlation function corresponds to the density profile of the Fermi-sea. Figure 4.4 shows the plot of $\hat{R}_1(-1,+1,\hat{x})$ for different values of $\hat{c}$, where $\hat{K}_1=-\hat{K}_2=\pi/8$ are chosen symmetrically around zero. While for $\hat{c}=0$ the density profile of the Fermi-sea is a constant determined by Eq. (4.31), it changes if the interaction is switched on. The density at the positions of the two spin-up particles decreases as the interaction increases and finally vanishes for $\hat{c}\to+\infty$. Figure 4.5 shows the same as Fig. 4.4 but for a higher value of $\hat{K}_1=-\hat{K}_2=\pi/4$ and slightly different values of the interaction strength. Comparing Fig. 4.4 with Fig. 4.5 reveals that with increasing $\hat{c}$ the suppression of the density in between the two spin-up particles is considerably stronger for higher values of $\hat{K}_1$ and $\hat{K}_2$. A qualitatively similar picture

emerges when $K_1$ and $K_2$ are inside the Fermi-sea but chosen in a non-symmetric way. The situation changes rather drastically if $\hat{K}_1$ is set to unity and $\hat{K}_2 = 0$ is in the core of the Fermi-sea. Figure 4.6 shows the corresponding plots. Now the density between the two spin-up particles is enhanced with respect to the density in the outer regions. With increasing interaction strength also the density inside the interval $(\hat{y}_1, \hat{y}_2)$ increases and finally for $\hat{c} \to \infty$ diverges in accordance with the discussion following Eq. (4.19). This reveals that the density profile of the Fermi-sea might be strongly influenced due to the presence of the two spin-up particles.

## 4.4 Summary of Part I

We studied the Yang-Gaudin model of spin one-half Fermions with repulsive contact interaction in the highly imbalanced limit.

The exact eigenfunctions can be constructed by means of the Bethe-Ansatz. However, the original form of the Bethe-Ansatz wave function turns out to be inconvenient for calculations where the eigenfunctions enter explicitly. Their exact reformulation, stated in Theorem 1, is the main result of Chapter 2. Using the novel form of the eigenfunctions, we re-derived the Bethe-Ansatz equations which determine the spectrum of the system for periodic boundary conditions. The thermodynamic limit has been discussed with special emphasis on the imbalanced case, where of one species only very few particles are present. We closed Chapter 2 by a comparison of the Yang-Gaudin model with its discretized counterpart, the one-dimensional Hubbard model. In particular we showed that, if only a single particle of one species but an arbitrary number of the other is present, the corresponding eigenfunctions can be cast into an analog form as for the continuous case. This leads to the conjecture that the reformulation of the exact eigenfunctions found for the Yang-Gaudin model also applies to the one-dimensional Hubbard model through a simple substitution of parameters.

In Chapter 3, we considered a single spin-up particle that interacts via a repulsive $\delta$-potential with a Fermi-sea. According to the result of Chapter 2, the many-body wave function can be expressed by a determinant. This simplifies considerably the calculations, since it allows to employ the tools of matrix algebra. By means of the reformulated eigenfunctions we were able to find the exact thermodynamic expressions for several quantities of the model.

As a function of the interaction strength and the extra particle's free momentum, we calculated the expectation values for the energy shift due to the interaction, the mean interaction energy as well as the kinetic energy of the extra particle. The results reveal, that if its momentum is located right at the Fermi-edge, the extra particle behaves like an additional non-interacting Fermion of the sea inasmuch as the energy shift and the interaction energy vanish for arbitrary interaction strength. From the energy shift due to the interaction we deduced the effective mass and the self-energy of the extra particle in the presence of the Fermi-sea. Furthermore we derived the density-density correlation function. This generalizes the previous result found by McGuire [4] to the case where the extra particle can occupy an arbitrary state within the Fermi-sea and moreover allowed

us to calculate the mean interaction energy for exited states of the extra particle.

Furthermore we studied the equal time single particle Green's function $G(y, y')$ of the extra particle. The determinantal form of the wave function permits $G(y, y')$ for an arbitrary number $N$ of spin-down particles to be expressed in terms of a single $N \times N$ determinant (see Eq. (3.46)). In the limit of infinitely strong interaction strength we were able to further evaluate the corresponding expression and to take the thermodynamic limit. The crucial observation was, that in the hardcore limit the Green's function is expressible in terms of solutions of Painlevé non-linear differential equations. In this framework the momentum of the extra particle enters via the short distance boundary conditions which characterize the solution. The merit of formulating the Green's function through solutions of Painlevé equations is twofold: On the one hand, the thermodynamic limit can be taken by simply going from Painlevé VI to Painlevé V. On the other hand, the existence of connection formulas permits to relate the short distance characteristics to the asymptotic behavior of the solution. Applied to the extra particle's Green's function, this allowed us to first, do thermodynamic limit, and second, to deduce an algebraic decay of $G(y, y')$ for large distances $x = y - y'$ as $G(x) \sim x^{-\alpha}$ with $\alpha = (\hat{K}^2 + 1)/2$, where $\hat{K} = K/k_{\mathrm{F}}$ is the extra particle's momentum measured on the scale of the Fermi-momentum. In that sense, the particle undergoes a transition from a free Fermion ($\hat{K} = 1$) to a hardcore Boson ($\hat{K} = 0$) as its momentum moves from the edge to the core of the Fermi-sea.

Up to now we were unable to take the thermodynamic limit of the full Green's function for finite interaction strength. Nevertheless, with the determinantal structure of the Green's function presented here, an analytical way to perform this limit is not out of reach. In particular it allowed us to expand the Green's function and thereby to deduce the exact short distance behavior for arbitrary interaction strength in the thermodynamic limit. The result showed that, the closer the extra particle's momentum is located to the Fermi-edge, the faster the Green's function decays for short distances. In combination with our numerical simulations and the analytical result found in the Tonks-Girardeau regime, this seems to indicate that a finite interaction strength does not change the Green's function quantitatively but rather renormalizes the parameters. One might conjecture that the relation of the Green's function to solutions of Painlevé equations holds beyond the hardcore limit. Such a relation indeed exists for hardcore Bosons and was employed [92, 85] to calculate the first order corrections in $1/c$ to the hardcore result (see Eq. (3.68)).

Regarding dynamical quantities, we considered the survival probability amplitude $S(t)$ and the closely related local density of states. In particular we discussed the scenario where the interaction is instantaneously switched on for $t > 0$. Since the initial state is not an eigenstate of the interacting system, the survival probability is expected to decay. The perturbative evaluation of $|S(t)|^2$ however, showed that this is the case only if the extra particle's momentum is located outside the Fermi-sea, predicting an infinitely slow decay of the initial state if the momentum of the extra particle is inside the Fermi-sea. Despite the fact that this phenomenon can be understood within the perturbative approach as a consequence of conservation of energy and momentum, it

can qualitatively not be true, since in any case the initial state is not an eigenstate of the interacting Hamiltonian. This clearly shows the limited applicability of perturbation theory in the present context. A non-perturbative evaluation of the survival probability or likewise the local density of states is a challenging task. We calculated the overlap integral between the initial state and the interacting eigenfunctions. This might be a first step towards that direction.

In Chapter 4, we studied the case where two minority Fermions are present. A detailed study of the system's total energy and the resulting energy shift due to the interaction allowed us to identify contributions which have a natural interpretation as effective interaction energy for the two minority particles.

By means of the reformulated eigenfunctions, we were able to calculate the two and three particle density-density correlation function for the two spin-up particles and the two spin-up particles with a particle of the Fermi-sea, respectively. Both quantities can be expressed in a unified way. As the interaction strength varies from zero to infinity, the two particle density-density correlation function undergoes a transition from the one for two free Fermions to that of a non-interacting Fermi-sea. Our result for the three particle density-density correlation function shows how the density profile of the Fermi-sea is affected due to the interaction with the two minority particles. The way in which this happens crucially depends on the values of the free momenta $K_1$ and $K_2$.

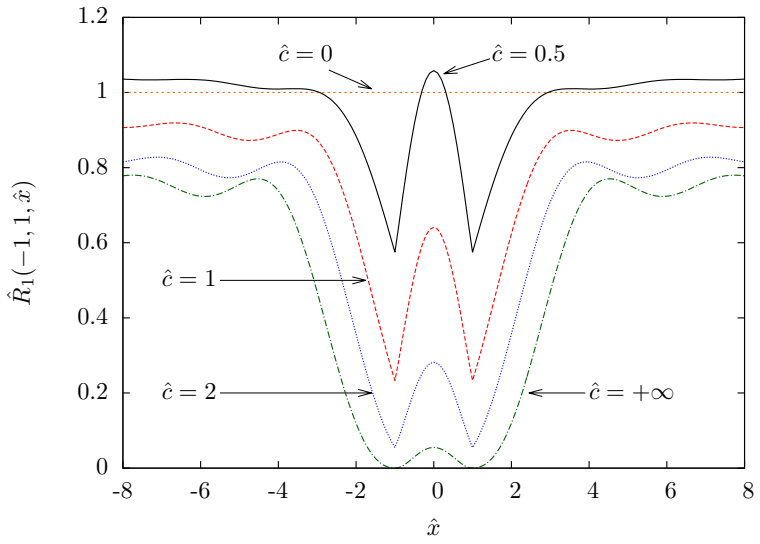


Figure 4.4: Three particle density-density correlation function for fixed $\hat{y}_1 = -1$ and $\hat{y}_2 = +1$ as function of $\hat{x}$ for $\hat{K}_1 = -\hat{K}_2 = \pi/8$. The values for the interaction strength are: $\hat{c} = 0$ (short dashed line, orange), $\hat{c} = 0.5$ (solid line, black), $\hat{c} = 1$ (dashed line, red), $\hat{c} = 2$ (dotted line, blue) and $\hat{c} = +\infty$ (dot-dashed line, green).

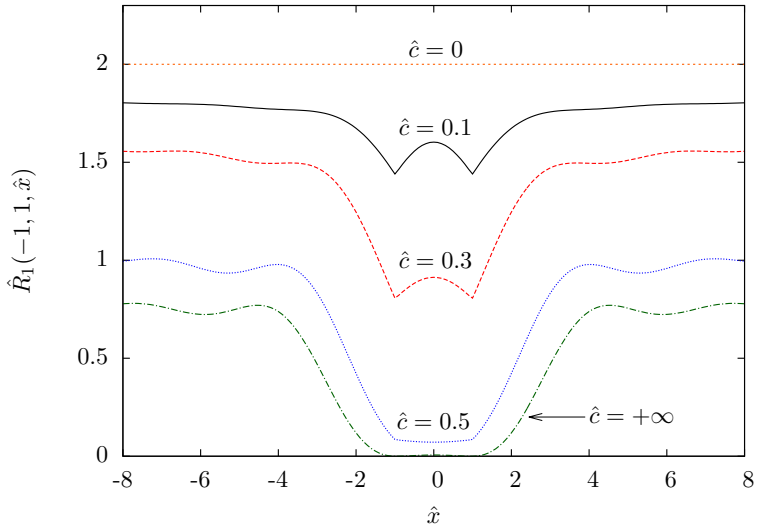


Figure 4.5: Three particle density-density correlation function for fixed $\hat{y}_1 = -1$ and $\hat{y}_2 = +1$ as function of $\hat{x}$ for the quantum numbers $\hat{K}_1 = -\hat{K}_2 = \pi/4$. The values for the interaction strength are: $\hat{c} = 0$ (short dashed line, orange), $\hat{c} = 0.1$ (solid line, black), $\hat{c} = 0.3$ (dashed line, red), $\hat{c} = 0.5$ (dotted line, blue) and $\hat{c} = +\infty$ (dot-dashed line, green).

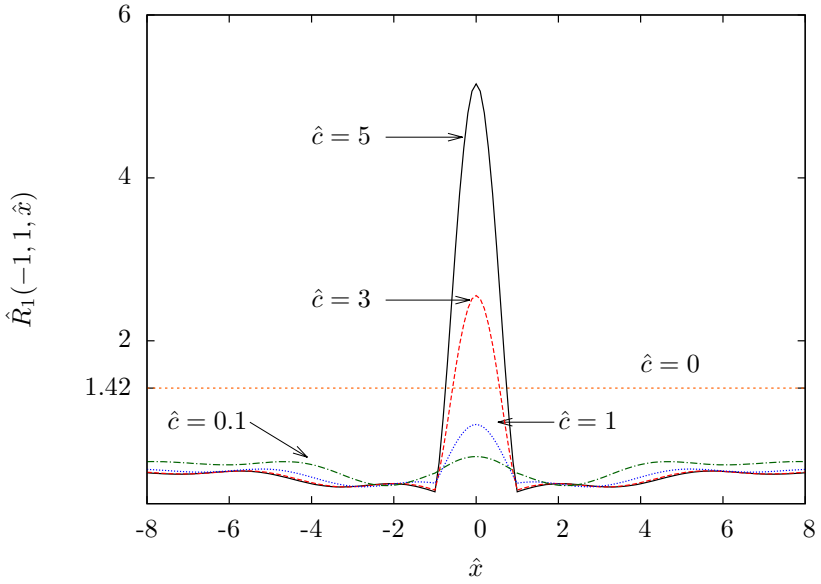


Figure 4.6: Three particle density-density correlation function for fixed $\hat{y}_1 = -1$ and $\hat{y}_2 = +1$ as function of $\hat{x}$ for the quantum numbers $\hat{K}_1 = 1$ and $\hat{K}_2 = 0$ . The values for the interaction strength are: $\hat{c} = 0$ (short dashed line, orange), $\hat{c} = 5$ (solid line, black), $\hat{c} = 3$ (dashed line, red), $\hat{c} = 1$ (dotted line, blue) and $\hat{c} = 0.1$ (dot-dashed line, green).

# Part II

# Creation and annihilation operators for spin one-half Fermions

# Chapter 5

# Exact diagonalization of spin one-half Fermions

In this chapter, we study one-dimensional systems with two sorts of Fermions. In particular the particles of each species are allowed to have distinct masses and to interact through a pairwise interaction potential with different coupling constants. Within this setup, we address the following question: What are the necessary conditions on the interaction potential, the masses and the coupling constants in order for the system to provide exact solutions? To address this problem we use an approach based on the explicit construction of particle creation and annihilation operators. For identical Fermions this method has been developed in Ref. [67]. Its extension to systems of spin one-half Fermions is the main result of this chapter. The outline is as follows:

In Sec. 5.1, we present the main idea of the approach as developed in Ref. [67] for the case of identical Fermions. The general properties of fermionic creation and annihilation operators are discussed in Sec. 5.2. Section 5.3 is devoted to their explicit construction. In Sec. 5.4, we discuss applications. We summarize and conclude in Sec. 5.5.

## 5.1 Identical Fermions

The essence of the approach [67] roots on the observation that a class of multidimensional integration formulas can be interpreted as particle creation operators in coordinate representation. Here we give a brief review on the corresponding method.

Our starting point is the Hamiltonian for $N$ identical Fermions in one dimension which interact through a pairwise interaction potential. Choosing the units such that $\hbar = 1$ and setting all masses equal to $1/2$ the Hamiltonian reads

$$H_N(\mathbf{x}) = -\sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} + \sum_{n \neq j}^{N} V(x_n - x_j) \ . \tag{5.1}$$

The properties of particle creation and annihilation operators for such a system are as follows:

Consider an eigenfunction $|\Psi_N(\mathbf{k})\rangle$ of the $N$ particle Hamiltonian $H_N$. The action of the creation operator $\hat{c}_k^\dagger$ onto $|\Psi_N(\mathbf{k})\rangle$ yields an eigenfunction $|\Psi_{N+1}(\mathbf{k}')\rangle$ of the $N+1$ particle Hamiltonian $H_{N+1}$. Analogously, the action of the annihilation operator $\hat{c}_k$ leads from $|\Psi_N(\mathbf{k})\rangle$ to an eigenfunction $|\Psi_{N-1}(\mathbf{k}'')\rangle$ of the Hamiltonian $H_{N-1}$. These relations read

$$|\Psi_{N+1}(\mathbf{k}')\rangle = \hat{c}_k^\dagger |\Psi_N(\mathbf{k})\rangle \qquad , \qquad |\Psi_{N-1}(\mathbf{k}'')\rangle = \hat{c}_k |\Psi_N(\mathbf{k})\rangle \ . \tag{5.2}$$

Here the operator $\hat{c}_k^{(\dagger)}$ annihilates (creates) a particle with momentum $k$ and $\mathbf{k}'$ and $\mathbf{k}''$ denote sets of $N \pm 1$ quasi-momenta, respectively. By means of the creation operator, the eigenfunctions of $H_N$ for an arbitrary number $N$ of particles can be constructed recursively.

Exemplarily we consider the creation operator. Its action onto $|\Psi_N(\mathbf{k})\rangle$ translates in coordinate representation to a multiple integral that relates the $N$ particle wave function $\Psi_N(\mathbf{x}, \mathbf{k})$ to the $N+1$ particle wave function $\Psi_{N+1}(\mathbf{x}', \mathbf{k}')$. Following Ref. [67] the first relation in Eq. (5.2) can be written as

$$\Psi_{N+1}(\mathbf{x}', \mathbf{k}') \propto \int_{x_2'}^{x_1'} dx_1 \int_{x_3'}^{x_2'} dx_2 \cdots \int_{x_{N+1}'}^{x_N'} dx_N c_k^\dagger(\mathbf{x}', \mathbf{x}) \Psi_N(\mathbf{x}, \mathbf{k}) \ . \tag{5.3}$$

The kernel $c_k^\dagger(\mathbf{x}', \mathbf{x})$ of this integral operator relates the two sets of coordinates $\mathbf{x}$ and $\mathbf{x}'$. We refer to $c_k^\dagger(\mathbf{x}', \mathbf{x})$ as creation function. As introduced above it is symmetric in both sets of arguments $\mathbf{x}'$ and $\mathbf{x}$. The antisymmetry of $\Psi_N(\mathbf{x}, \mathbf{k})$ on the right hand side ensures together with the boundaries of the integral the antisymmetry of $\Psi_{N+1}(\mathbf{x}', \mathbf{k}')$ on the left hand side. In particular the boundaries induce the ordering

$$x_1' > x_1 > x_2' > x_2 > \cdots > x_N' > x_N > x_{N+1}' \tag{5.4}$$

between the primed and the unprimed variables. The essential ingredient in formula (5.3) is the integration kernel $c_k^\dagger(\mathbf{x}', \mathbf{x})$. The main result of Ref. [67] consists in the explicit construction of $c_k^\dagger(\mathbf{x}', \mathbf{x})$ and the corresponding function $c_k(\mathbf{x}'', \mathbf{x})$ for the annihilation operator for a series of two-body potentials. We state the explicit form of $c_k^\dagger(\mathbf{x}', \mathbf{x})$ for two kinds of potentials.

For contact interaction we write the potential in the Hamiltonian as $V(x) = -2c\delta(x)$, where the coupling constant $c < 0$ is negative (repulsive interaction). According to Ref. [67], the creation function acquires in this case the form

$$\begin{aligned} c_k^\dagger(\mathbf{x}', \mathbf{x}) \quad = \quad & \exp\left( ik \left( \sum_{n=1}^{N+1} x_n' - \sum_{m=1}^{N} x_m \right) \right) \\ & \exp\left( -c \sum_{n<m}^{N+1} |x_n' - x_m'| + c \sum_{n=1}^{N+1} \sum_{m=1}^{N} |x_n' - x_m| - c \sum_{n<m}^{N} |x_n - x_m| \right) \ . \end{aligned} \tag{5.5}$$

In a self-evident way, the creation function factorizes into a part that accounts for the interaction and a non-interacting part. This structure of the creation function is a common feature that holds for all interaction potentials that can be treated within the present approach. Now it is a well-known fact that any local potential is invisible for identical Fermions. In the present context this is reflected as follows: Substitution of $c_k^\dagger(\mathbf{x}', \mathbf{x})$ into Eq. (5.3), the boundaries of the integral impose the ordering (5.4) between the primed and the unprimed variables. It is then easily seen that the absolute values in Eq. (5.5) can be eliminated and that the exponent of the second exponential vanishes identically. Thus for identical Fermions the creation function (5.5) is equivalent to the non-interacting one. This demonstrates the intimate relation between the boundaries of the multiple integral (5.3) and the statistics of the particles.

Two further examples are the inverse square potential $V(x) = \lambda(\lambda+1)/x^2$ and its periodic version $V(x) = b^2\lambda(\lambda+1)/\sin^2(bx)$. The parameters $\lambda > -1$ and $b$ determine the interaction strength and the periodicity, respectively. For the inverse square potential the creation function acquires the form

$$c_k^\dagger(\mathbf{x}', \mathbf{x}) = \exp\left(ik\left(\sum_{n=1}^{N+1} x_n' - \sum_{m=1}^{N} x_m\right)\right) \frac{\prod_{n=1}^{N+1}\prod_{m=1}^{N} |x_n' - x_m|^\lambda}{\prod_{n<m}^{N+1} |x_n' - x_m'|^\lambda \prod_{n<m}^{N} |x_n - x_m|^\lambda} . \quad (5.6)$$

A similar form of the creation function is relevant for the recursive construction of certain group integrals [93, 94]. In particular the inverse square potential has been studied in the context of particles with different masses [95, 96, 97, 98] and it has been shown that under certain conditions for the masses, the eigenfunctions can be constructed recursively [97, 98].

The creation function for the periodic model is obtained from Eq. (5.6) by substituting $|x| \to |\sin(bx)|$. The eigenfunctions are determined by a class of symmetric polynomials, the Jack-polynomials [64]. Substituting $c_k^\dagger(\mathbf{x}', \mathbf{x})$ into Eq. (5.3), the resulting expression has been identified with a recursive integral representation of those [67].

To what extend two-component mixtures with different masses for other interaction potentials are exactly solvable is an open question. In the following we shall address this question by extending the above sketched approach to a fermionic two component mixture with unequal masses.

## 5.2 Spin one-half Fermions in one dimension

We consider non-relativistic spin one-half Fermions in one dimension which interact via a two body potential $V(x)$. In second quantization the Hamiltonian reads

$$\hat{H} = \frac{1}{2m_1}\int_\Omega dx \hat{\Psi}_\downarrow^\dagger(x)\left(-\frac{d^2}{dx^2}\right)\hat{\Psi}_\downarrow(x) + \frac{1}{2m_2}\int_\Omega dy \hat{\Psi}_\uparrow^\dagger(y)\left(-\frac{d^2}{dy^2}\right)\hat{\Psi}_\uparrow(y) \quad (5.7)$$

$$
\begin{aligned}
&+g_1 \int\limits_{\Omega} dx \int\limits_{\Omega} dx' \hat{\Psi}_{\downarrow}^{\dagger}(x)\hat{\Psi}_{\downarrow}^{\dagger}(x')V(x-x')\hat{\Psi}_{\downarrow}(x')\hat{\Psi}_{\downarrow}(x) \\
&+g_2 \int\limits_{\Omega} dy \int\limits_{\Omega} dy' \hat{\Psi}_{\uparrow}^{\dagger}(y)\hat{\Psi}_{\uparrow}^{\dagger}(y')V(y-y')\hat{\Psi}_{\uparrow}(y')\hat{\Psi}_{\uparrow}(y) \\
&+g_3 \int\limits_{\Omega} dx \int\limits_{\Omega} dy \hat{\Psi}_{\downarrow}^{\dagger}(x)\hat{\Psi}_{\uparrow}^{\dagger}(y)V(x-y)\hat{\Psi}_{\uparrow}(y)\hat{\Psi}_{\downarrow}(x) \ ,
\end{aligned}
$$

where $m_1$ and $m_2$ are the masses of the spin-down and the spin-up particles, respectively. The coupling constants for the interaction between the different particle species are denoted by $g_1$, $g_2$ and $g_3$. In a general setup the masses as well as the coupling constants might assumed to be different i.e. $m_1 \neq m_2$ and $g_1 \neq g_2 \neq g_3$. However, we emphasize that the functional form $V(x)$ of the interaction potential is the same for all particle species.

In Eq. (5.7), the units are chosen such that $\hbar = 1$. The masses as well as the coupling constants are assumed to be dimensionless. The integration domain $\Omega$ is either the real axis for a scattering system i.e. $\Omega = \mathbb{R}$ or a compact interval for periodic boundary conditions. Furthermore, $\hat{\Psi}_{\uparrow,\downarrow}^{(\dagger)}$ are fermionic field operators. They fulfill the anticommutator relations

$$
\{\hat{\Psi}_{\sigma}(x), \hat{\Psi}_{\sigma'}^{\dagger}(x')\} = \delta(x-x')\delta_{\sigma\sigma'} \quad , \quad \{\hat{\Psi}_{\sigma}^{(\dagger)}(x), \hat{\Psi}_{\sigma'}^{(\dagger)}(x')\} = 0 \ , \tag{5.8}
$$

where $\sigma = \uparrow, \downarrow$ stands for the two possible spin-polarizations spin-up and spin-down and $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ denotes the anticommutator of two operators $\hat{A}$ and $\hat{B}$.

We aim at mapping the Hamiltonian (5.7) onto the quadratic form

$$
\hat{H} = \sum_{\sigma=\uparrow,\downarrow} \sum_{k} \epsilon_{\sigma}(k)\hat{a}_{\sigma k}^{\dagger}\hat{a}_{\sigma k} \ . \tag{5.9}
$$

Here the operators $\hat{a}_{\sigma k}^{(\dagger)}$ annihilate (create) a particle with momentum $k$ and spin $\sigma$, and $\epsilon_{\sigma}(k)$ is the energy of the annihilated (created) particle. Formally the equation above corresponds to the Hamiltonian of non-interacting spin one-half Fermions. However, the interaction is encoded in the annihilation and creation operators and the spectrum of the system, which differ from those of free particles. The anticommutator relations (5.8) for the field operators carry over to $\hat{a}_{\sigma k}$ and $\hat{a}_{\sigma k}^{\dagger}$

$$
\{\hat{a}_{\sigma k}, \hat{a}_{\sigma' k'}^{\dagger}\} = \delta_{kk'}\delta_{\sigma\sigma'} \quad , \quad \{\hat{a}_{\sigma k}^{(\dagger)}, \hat{a}_{\sigma' k'}^{(\dagger)}\} = 0 \ . \tag{5.10}
$$

Acting with the Hamiltonian (5.9) onto $\hat{a}_{\sigma k}^{(\dagger)}$ and using the anticommuting properties above yields

$$
\left[\hat{H}, \hat{a}_{\sigma k}^{\dagger}\right] = \epsilon_{\sigma}(k)\hat{a}_{\sigma k}^{\dagger} \ , \tag{5.11}
$$

$$
\left[\hat{H}, \hat{a}_{\sigma k}\right] = -\epsilon_{\sigma}(k)\hat{a}_{\sigma k} \ , \tag{5.12}
$$

where we use the notation $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ for the commutator of two operators. The commutator relations (5.11) and (5.12), together with the Hamiltonian define the creation and annihilation operators. They will serve as starting point for the explicit construction of $\hat{a}^\dagger_{\sigma k}$ and $\hat{a}^\dagger_{\sigma k}$ in coordinate representation.

By means of the creation operator the eigenfunctions of the Hamiltonian for an arbitrary number of spin-up and spin-down particles can be obtained by successive application of $\hat{a}^\dagger_{\uparrow k}$ and $\hat{a}^\dagger_{\downarrow k}$ on the vacuum. In this way the eigenstates for $N$ spin-down and $M$ spin-up particles can be constructed by

$$|\Psi_{N,M}(\mathbf{k})\rangle = \prod_{n=M+1}^{N} \hat{a}^\dagger_{\downarrow k_n} \prod_{m=1}^{M} \hat{a}^\dagger_{\uparrow k_m} |0\rangle \ , \tag{5.13}$$

where $|0\rangle$ denotes the vacuum state i.e. the state where no particle is present. The set $\mathbf{k} = \{k_1, \ldots, k_M, k_{M+1}, \ldots, k_{N+M}\}$ comprises the quasi-momenta of the $N+M$ particle state and is organized such that the first $M$ elements account for the quasi-momenta of the spin-up Fermions, whereas the last $N$ elements are associated with the momenta of the spin-down particles. The states (5.13) are eigenstates to the $N+M$ particle Hamiltonian $\hat{H}_{N,M}$ such that

$$\hat{H}_{N,M}|\Psi_{N,M}(\mathbf{k})\rangle = \left( \sum_{m=1}^{M} \epsilon_\uparrow(k_m) + \sum_{n=1}^{N} \epsilon_\downarrow(k_{M+n}) \right) |\Psi_{N,M}(\mathbf{k})\rangle \ . \tag{5.14}$$

Acting with the annihilation (creation) operator $\hat{a}^{(\dagger)}_{\downarrow k}$ onto an eigenfunction $|\Psi_{N,M}(\mathbf{k})\rangle$ of $\hat{H}_{N,M}$ yields an eigenfunction of $\hat{H}_{N-1,M}$ ($\hat{H}_{N+1,M}$), that is the wave function with the number of spin-down particles decreased (increased) by one

$$|\Psi_{N-1,M}(\mathbf{k}'')\rangle = \hat{a}_{\downarrow k}|\Psi_{N,M}(\mathbf{k})\rangle \qquad , \qquad |\Psi_{N+1,M}(\mathbf{k}')\rangle = \hat{a}^\dagger_{\downarrow k}|\Psi_{N,M}(\mathbf{k})\rangle \ . \tag{5.15}$$

The sets of quasi-momenta $\mathbf{k}''$ and $\mathbf{k}'$ are obtained from $\mathbf{k}$ by dropping or appending the quasi-momentum $k$, respectively. Analogously the action of $\hat{a}^{(\dagger)}_{\uparrow k}$ onto $|\Psi_{N,M}(\mathbf{k})\rangle$ yields eigenstates of $\hat{H}_{N,M-1}$ ($\hat{H}_{N,M+1}$). In the squeal we discuss the first quantized form of the quantities introduced above in coordinate representation.

## 5.3 Representation in coordinate space

For a fixed number of particles the Hamiltonian (5.7) can be written in first quantized form. In coordinate representation we have for $N$ spin-down and $M$ spin-up particles

$$\begin{aligned} H_{N,M}(\mathbf{z}) &= -\frac{1}{2m_1}\sum_{n=1}^{N}\frac{\partial^2}{\partial x_n^2} - \frac{1}{2m_2}\sum_{m=1}^{M}\frac{\partial^2}{\partial y_m^2} \\ &+ g_1\sum_{i\neq j}^{N} V(x_i - x_j) + g_2\sum_{i\neq j}^{M} V(y_i - y_j) + g_3\sum_{i=1}^{N}\sum_{j=1}^{M} V(x_i - y_j) \ , \end{aligned} \tag{5.16}$$

where the sets $\mathbf{x} = \{x_n\}_{n=1,\ldots,N}$ and $\mathbf{y} = \{y_m\}_{m=1,\ldots,M}$ comprise the coordinates of the spin-down and the spin-up Fermions, respectively and $\mathbf{z} = \{\mathbf{x}, \mathbf{y}\}$. Since it is convenient for the following discussion, we also introduce the coordinate sets where the number of spin-down particles is increased or decreased by one

$$\begin{aligned} \mathbf{x}' = \{x'_n\}_{n=1,\ldots,N+1} \quad , \quad \mathbf{y}' &= \{y'_m\}_{m=1,\ldots,M} \quad , \quad \mathbf{z}' = \{\mathbf{x}', \mathbf{y}'\} \ , \\ \mathbf{x}'' = \{x''_n\}_{n=1,\ldots,N-1} \quad , \quad \mathbf{y}'' &= \{y''_m\}_{m=1,\ldots,M} \quad , \quad \mathbf{z}'' = \{\mathbf{x}'', \mathbf{y}''\} \ . \end{aligned} \tag{5.17}$$

We require the system described by the Hamiltonian (5.7) to remain invariant if the spin-up and spin-down particles are exchanged. Therefore it is sufficient to restrict the discussion to one of the particle species. We consider the case of spin-down Fermions. Thus, we drop from now on the index $\sigma$ and write $\hat{a}_k^{(\dagger)} \equiv \hat{a}_{\downarrow k}^{(\dagger)}$ to unburden the notation.

The $N + M$ particle wave function in coordinate space $\Psi_{N,M}(\mathbf{z}, \mathbf{k}) = \langle \mathbf{z} | \Psi_{N,M}(\mathbf{k}) \rangle$ depends on the particle positions $\mathbf{x}$ and $\mathbf{y}$ of the spin-down and the spin-up Fermions as well as on the quasi-momenta $\mathbf{k}$. Since we are dealing with spin one-half Fermions, we require the wave function to be completely antisymmetric in the two sets of coordinates $\mathbf{x}$ and $\mathbf{y}$. Furthermore, $\Psi_{N,M}(\mathbf{z}, \mathbf{k})$ has to be antisymmetric in the partial sets $k_1, \ldots, k_M$ and $k_{M+1}, \ldots, k_{N+M}$, which correspond to the quasi-momenta for the spin-up and the spin-down particles, respectively.

We consider the action of $\hat{a}_k^\dagger$ onto the $N + M$ particle state $|\Psi_{N,M}(\mathbf{k})\rangle$ in coordinate representation. Multiplying the corresponding relation in Eq. (5.15) from the left with $\langle \mathbf{z}' |$ and inserting a complete set of $N + M$ particle states $|\mathbf{z}\rangle$, we obtain

$$\begin{aligned} \langle \mathbf{z}' | \Psi_{N+1,M}(\mathbf{k}') \rangle &= \langle \mathbf{z}' | \hat{a}_k | \Psi_{N,M}(\mathbf{k}) \rangle = \int\limits_{\Omega^{N+M}} d[\mathbf{z}] \langle \mathbf{z}' | \hat{a}_k | \mathbf{z} \rangle \langle \mathbf{z} | \Psi_{N,M}(\mathbf{k}) \rangle \\ \Longleftrightarrow \qquad \Psi_{N+1,M}(\mathbf{z}', \mathbf{k}') &= \int\limits_{\Omega^{N+M}} d[\mathbf{z}] \bar{a}_k^\dagger(\mathbf{z}', \mathbf{z}) \Psi_{N,M}(\mathbf{z}, \mathbf{k}) \ , \end{aligned} \tag{5.18}$$

where the notation $\bar{a}_k^\dagger(\mathbf{z}', \mathbf{z}) = \langle \mathbf{z}' | \hat{a}_k^\dagger | \mathbf{z} \rangle$ and the volume element $d[\mathbf{z}] = dx_1, \ldots, dx_N \ dy_1 \cdots dy_M$ have been introduced. Similarly it is found that the action of the annihilation operator onto $|\Psi_{N,M}(\mathbf{k})\rangle$ has the coordinate representation

$$\Psi_{N-1,M}(\mathbf{z}'', \mathbf{k}'') = \int\limits_{\Omega^{N+M}} d[\mathbf{z}] \bar{a}_k(\mathbf{z}'', \mathbf{z}) \Psi_{N,M}(\mathbf{z}, \mathbf{k}) \tag{5.19}$$

with $\bar{a}_k(\mathbf{z}'', \mathbf{z}) = \langle \mathbf{z}'' | \hat{a}_k | \mathbf{z} \rangle$. Equations (5.18) and (5.19) reveal the nature of $\hat{a}_k^\dagger$ and $\hat{a}_k$ as integral operators in configuration space. The general scheme of these relations is as follows: The wave function with the number of spin-down particles increased or decreased by one is obtained by integration over the original wave function with a corresponding integral kernel. Given an explicit expression for $\bar{a}_k^\dagger(\mathbf{z}', \mathbf{z})$, the eigenfunctions of the Hamiltonian (5.16) can be constructed recursively.

The symmetry of the wave function carries over to the creation and annihilation functions. This can be seen as follows: Consider the relation (5.18) for the creation operator.

Since it is fermionic, the wave function $\Psi_{N,M}(\mathbf{z},\mathbf{k})$ on the right hand side is completely antisymmetric in the coordinates $\mathbf{x}$ and $\mathbf{y}$. Therefore any symmetric contribution of $\bar{a}_k^\dagger(\mathbf{z}',\mathbf{z}) = \bar{a}_k^\dagger(\mathbf{x}',\mathbf{y}',\mathbf{x},\mathbf{y})$ in $\mathbf{x}$ and $\mathbf{y}$ drops out under the integral. Consequently the creation function can be taken as completely antisymmetric in the sets $\mathbf{x}$ and $\mathbf{y}$. On the other hand, we also demand the wave function $\Psi_{N+1,M}(\mathbf{z}',\mathbf{k}')$ on the left hand side of Eq. (5.18) to be fermionic. Thus the creation function must be completely antisymmetric in $\mathbf{x}'$ and $\mathbf{y}'$ as well. Analogously one argues to deduce antisymmetry of the annihilation function in the coordinates $\mathbf{x}''$, $\mathbf{y}''$, $\mathbf{x}$ and $\mathbf{y}$. We refer to $\bar{a}_k^{(\dagger)}(\mathbf{z}''^{(\prime)},\mathbf{z})$ also as *antisymmetric* annihilation (creation) function.

We discuss how the defining commutator relations (5.11) and (5.12) translate to configuration space. For the creation operator we multiply Eq. (5.11) from the left with $\langle\mathbf{z}'|$ and from the right with $|\Psi_{N,M}\rangle$. Then using the completeness and orthogonality relations, Eq. (5.11) can be cast into the form

$$0 \stackrel{!}{=} \int\limits_{\Omega^{N+M}} d[\mathbf{z}] \left[\Big(H_{N+1,M}(\mathbf{z}') - H_{N,M}(\mathbf{z}) - \epsilon_\downarrow(k)\Big)\bar{a}_k^\dagger(\mathbf{z}',\mathbf{z})\right] \Psi_{N,M}(\mathbf{z},\mathbf{k}) \ . \tag{5.20}$$

We consider the annihilation operator. Multiplying Eq. (5.12) from the left with $\langle\mathbf{z}''|$ and from the right with $|\Psi_{N,M}\rangle$ and again using the completeness and the orthogonality yields here

$$0 \stackrel{!}{=} \int\limits_{\Omega^{N+M}} d[\mathbf{z}] \left[\Big(H_{N-1,M}(\mathbf{z}'') - H_{N,M}(\mathbf{z}) + \epsilon_\downarrow(k)\Big)\bar{a}_k(\mathbf{z}'',\mathbf{z})\right] \Psi_{N,M}(\mathbf{z},\mathbf{k}) \ , \tag{5.21}$$

Equations (5.20) and (5.21) have to hold for any eigenfunction $\Psi_{N,M}(\mathbf{z},\mathbf{k})$. This can only be true if the terms in the square brackets vanish identically. The commutator relations (5.11) and (5.12) thus have the coordinate representation

$$\Big(H_{N+1,M}(\mathbf{z}') - H_{N,M}(\mathbf{z})\Big)\bar{a}_k^\dagger(\mathbf{z}',\mathbf{z}) = \epsilon_\downarrow(k)\bar{a}_k^\dagger(\mathbf{z}',\mathbf{z}) \ , \tag{5.22}$$

$$\Big(H_{N-1,M}(\mathbf{z}'') - H_{N,M}(\mathbf{z})\Big)\bar{a}_k(\mathbf{z}'',\mathbf{z}) = -\epsilon_\downarrow(k)\bar{a}_k(\mathbf{z}'',\mathbf{z}) \ . \tag{5.23}$$

This is a set of partial differential equations. Their solutions determine the antisymmetric creation and annihilation functions.

### 5.3.1 Statistical functions

It proves useful to split the creation and annihilation function into a part that covers the statistical features discussed above and into a purely symmetric part which accounts for the interaction. Following Ref. [67] we introduce the functions

$$I_{N\downarrow}^\dagger(\mathbf{x}',\mathbf{x}) = \frac{1}{2^{N+1}(N+1)!} \det\Big[\mathrm{sgn}(x_n' - x_l)\Big|1\Big]_{\substack{n=1,\ldots,N+1\\ l=1,\ldots,N}} \ , \tag{5.24}$$

$$I_{N\downarrow}(\mathbf{x}'',\mathbf{x}) = \frac{1}{2^N N!} \det\Big[\mathrm{sgn}(x_l - x_n'')\Big|1\Big]_{\substack{l=1,\ldots,N\\ n=1,\ldots,N-1}} \ , \tag{5.25}$$

which account for the statistics of the creation and the annihilation functions with respect to the spin-down particles. Due to their determinantal form, the expressions (5.24) and (5.25) are completely antisymmetric in both sets of arguments and thus by construction have the desired symmetry of $\bar{a}_k^\dagger(\mathbf{z}', \mathbf{z})$ and $\bar{a}_k(\mathbf{z}'', \mathbf{z})$.

The action of the creation and annihilation operators onto a wave function leaves the number of spin-up particles unchanged. This suggests to introduce the corresponding quantities here as follows

$$I_{M\uparrow}^\dagger(\mathbf{y}', \mathbf{y}) = \frac{1}{2^M M!} \det\left[\mathrm{sgn}(y'_m - y_l)\right]_{m,l=1,\ldots,M} , \tag{5.26}$$

$$I_{M\uparrow}(\mathbf{y}'', \mathbf{y}) = I_{M\uparrow}^\dagger(\mathbf{y}, \mathbf{y}'') . \tag{5.27}$$

We refer to the expressions in Eqs. (5.24)-(5.27) also as statistical functions. Using them, the statistics encoded in the antisymmetric creation and annihilation function can separated through the Ansatz

$$\bar{a}_k^\dagger(\mathbf{z}', \mathbf{z}) = I_{N\downarrow}^\dagger(\mathbf{x}', \mathbf{x}) I_{M\uparrow}^\dagger(\mathbf{y}', \mathbf{y}) a_k^\dagger(\mathbf{z}', \mathbf{z}) , \tag{5.28}$$

$$\bar{a}_k(\mathbf{z}'', \mathbf{z}) = I_{N\downarrow}(\mathbf{x}'', \mathbf{x}) I_{M\uparrow}(\mathbf{y}'', \mathbf{y}) a_k(\mathbf{z}'', \mathbf{z}) , \tag{5.29}$$

where $a_k^\dagger(\mathbf{z}', \mathbf{z})$ and $a_k(\mathbf{z}'', \mathbf{z})$ are now symmetric in all sets of variables. We refer to them as *symmetric* creation and annihilation functions, respectively.

The statistical functions above apply for scattering systems. To define their periodic counterparts we follow Ref. [67] and introduce the sawtooth function

$$[x] = x - nL \quad , \quad n = max(m \in \mathbb{Z} | m \leq x) . \tag{5.30}$$

The statistical functions with period $L$ are then obtained from Eqs. (5.24)-(5.27) by replacing the arguments according to

$$I_{N\downarrow}^{(\dagger)}(\mathbf{x}''^{(\prime)}, \mathbf{x}) \longrightarrow I_{N\downarrow}^{(\dagger)}([\mathbf{x}''^{(\prime)}], [\mathbf{x}]) \qquad , \qquad I_{M\uparrow}^{(\dagger)}(\mathbf{y}''^{(\prime)}, \mathbf{y}) \longrightarrow I_{M\uparrow}^{(\dagger)}([\mathbf{y}''^{(\prime)}], [\mathbf{y}]) . \tag{5.31}$$

The merit of the Ansatz (5.28) and (5.29) is that it separates the statistics imposed by the symmetry of wave function from the part that accounts for the interaction. Furthermore the specific form of the statistical functions allows to generalize the approach from Fermions to particles with other symmetries. For Bosons this is achieved by substituting the determinants by permanents. The permanent of a $N \times N$ matrix $A = [A_{jl}]$ $j, l = 1, \ldots, N$ is defined by

$$\mathrm{perm}(A) = \sum_{Q \in S(N)} \prod_{j=1}^{N} A_{jQ(j)} . \tag{5.32}$$

The only difference to the definition of the determinant is that the alternating sign for even and odd permutations is dropped. The consequence is that the permanent of the matrix $A$ is symmetric when two columns or two rows are exchanged [1]. Thus replacing the determinants in Eqs. (5.24)-(5.27) by permanents, the resulting expressions are *symmetric* with respect to the exchange of two coordinates and thus by construction have the nature of a bosonic wave function.

[1] This implies that the permanent does not vanish when two columns or rows coincide

### 5.3.2 Symmetric creation and annihilation functions

We discuss the symmetric creation and annihilation functions $a_k^\dagger(\mathbf{z}', \mathbf{z})$ and $a_k(\mathbf{z}'', \mathbf{z})$. The appearance of the statistical functions in Eq. (5.28) and (5.29) is a residual of the statistics imposed on the eigenfunctions. However, the Hamiltonian $H_{N\pm1,M}$ is independent of any statistical conditions claimed for its eigenfunctions. Hence we expect the symmetric creation and annihilation functions to fulfill the same differential equations as the antisymmetric ones. That is

$$\Big(H_{N+1,M}(\mathbf{z}') - H_{N,M}(\mathbf{z})\Big) a_k^\dagger(\mathbf{z}', \mathbf{z}) = \epsilon_\downarrow(k) a_k^\dagger(\mathbf{z}', \mathbf{z}) \ , \tag{5.33}$$

$$\Big(H_{N-1,M}(\mathbf{z}'') - H_{N,M}(\mathbf{z})\Big) a_k(\mathbf{z}'', \mathbf{z}) = -\epsilon_\downarrow(k) a_k(\mathbf{z}'', \mathbf{z}) \ . \tag{5.34}$$

For this to be true, we require the product of the statistical functions to commute with the difference $H_{N\pm1,M}(\mathbf{z}') - H_{N,M}(\mathbf{z})$ i.e.

$$\Big[H_{N+1,M}(\mathbf{z}') - H_{N,M}(\mathbf{z}), I_{N\downarrow}^\dagger(\mathbf{x}', \mathbf{x}) I_{M\uparrow}^\dagger(\mathbf{y}', \mathbf{y})\Big] a_k^\dagger(\mathbf{z}', \mathbf{z}) = 0 \ , \tag{5.35}$$

$$\Big[H_{N-1,M}(\mathbf{z}'') - H_{N,M}(\mathbf{z}), I_{N\downarrow}(\mathbf{x}'', \mathbf{x}) I_{M\uparrow}(\mathbf{y}'', \mathbf{y})\Big] a_k(\mathbf{z}'', \mathbf{z}) = 0 \ . \tag{5.36}$$

Using the explicit form of the symmetric creation and annihilation functions stated below, we proof that these conditions are indeed fulfilled in Sec. (B.1) of Appendix B.

We state the main result regarding the solutions of the differential equations (5.33) and (5.34) as theorem.

**Theorem 2** *1. Let $F(x) = F(-x)$ be a symmetric function whose first derivative $F'(x) \equiv f(x)$ is a solution of the functional equation*

$$f(x)f(y) + f(x)f(z) + f(y)f(z) = const. \ , \qquad x + y + z = 0 \ . \tag{5.37}$$

*Let furthermore $\lambda_1$, $\lambda_2$ and $\lambda_3$ be three real parameters which are chosen such that the conditions*

$$\frac{\lambda_3}{m_2} = \frac{\lambda_1}{m_1} \quad , \quad \frac{\lambda_3}{m_1} = \frac{\lambda_2}{m_2} \quad , \quad (m_1 - m_2)(\lambda_3 - 1) = 0 \tag{5.38}$$

*hold. Then solutions to the differential equations (5.33) and (5.34) for the symmetric creation and annihilation functions $a_k^\dagger(\boldsymbol{z}, \boldsymbol{z}')$ and $a_k(\boldsymbol{z}, \boldsymbol{z}')$ are given by*

$$\begin{aligned} a_k^\dagger(\boldsymbol{z}', \boldsymbol{z}) &= \exp\left(2\imath k m_1 \left(\sum_{n=1}^{N+1} x_n' - \sum_{m=1}^{N} x_m\right) + 2\imath k m_2 \sum_{m=1}^{M} \left(y_m' - y_m\right)\right. \\ &+ \lambda_1 \left(-\sum_{i<j}^{N+1} F(x_i' - x_j') + \sum_{i=1}^{N+1}\sum_{j=1}^{N} F(x_i' - x_j) - \sum_{i<j}^{N} F(x_i - x_j)\right) \\ &+ \left.\lambda_2 \left(-\sum_{i<j}^{M} F(y_i' - y_j') + \sum_{i=1}^{M}\sum_{j=1}^{M} F(y_i' - y_j) - \sum_{i<j}^{M} F(y_i - y_j)\right)\right. \end{aligned} \tag{5.39}$$

$$
\begin{aligned}
&+ \quad \lambda_3 \left( - \sum_{i=1}^{N+1} \sum_{m=1}^{M} (F(x_i' - y_j') - F(x_i' - y_j)) \right. \\
&\qquad \left. + \sum_{i=1}^{N} \sum_{m=1}^{M} (F(x_i - y_j') - F(x_i - y_j)) \right) \right) \ , \\
a_k(\boldsymbol{z}'', \boldsymbol{z}) \ &= \ \exp \left( 2\imath k m_1 \left( \sum_{n=1}^{N-1} x_n'' - \sum_{m=1}^{N} x_m \right) + 2\imath k m_2 \sum_{m=1}^{M} \left( y_m'' - y_m \right) \right. \qquad (5.40) \\
&+ \quad \lambda_1 \left( - \sum_{i<j}^{N-1} F(x_i'' - x_j'') + \sum_{i=1}^{N-1} \sum_{j=1}^{N} F(x_i'' - x_j) - \sum_{i<j}^{N} F(x_i - x_j) \right) \\
&+ \quad \lambda_2 \left( - \sum_{i<j}^{M} F(y_i'' - y_j'') + \sum_{i=1}^{M} \sum_{j=1}^{M} F(y_i'' - y_j) - \sum_{i<j}^{M} F(y_i - y_j) \right) \\
&+ \quad \lambda_3 \left( - \sum_{i=1}^{N-1} \sum_{m=1}^{M} (F(x_i'' - y_j'') - F(x_i'' - y_j)) \right. \\
&\qquad \left. \left. + \sum_{i=1}^{N} \sum_{m=1}^{M} (F(x_i - y_j'') - F(x_i - y_j)) \right) \right)
\end{aligned}
$$

*with the dispersion relation*

$$\epsilon_{\downarrow}(k) = 2m_1 k^2 \ . \qquad (5.41)$$

2. *The coupling constants $g_1$, $g_2$ and $g_3$ relate to the parameters $\lambda_1$, $\lambda_2$ and $\lambda_3$ and the masses $m_1$ and $m_2$ by*

$$g_1 = \frac{\lambda_1(\lambda_1+1)}{2m_1} \quad , \quad g_2 = \frac{\lambda_2(\lambda_2+1)}{2m_2} \quad , \quad g_3 = \frac{\lambda_3(\lambda_3+1)}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \ . \qquad (5.42)$$

*The two-body potential $V(x)$ in the Hamiltonian (5.16) is determined by the function $f(x)$ through*

$$V(x) = f^2(x) - f'(x) + const. \ . \qquad (5.43)$$

3. *The symmetric creation and annihilation functions fulfill*

$$
\begin{aligned}
\left[ P_{N+1,M}(\boldsymbol{z}') - P_{N,M}(\boldsymbol{z}) \right] a_k^{\dagger}(\boldsymbol{z}', \boldsymbol{z}) \ &= \ 2m_1 k a_k^{\dagger}(\boldsymbol{z}', \boldsymbol{z}) \ , \qquad (5.44) \\
\left[ P_{N-1,M}(\boldsymbol{z}'') - P_{N,M}(\boldsymbol{z}) \right] a_k(\boldsymbol{z}'', \boldsymbol{z}) \ &= \ -2m_1 k a_k(\boldsymbol{z}'', \boldsymbol{z}) \ , \qquad (5.45)
\end{aligned}
$$

*where*

$$P_{N,M}(\boldsymbol{z}) = \frac{1}{\imath} \left( \sum_{n=1}^{N} \frac{\partial}{\partial x_n} + \sum_{m=1}^{M} \frac{\partial}{\partial y_m} \right) \qquad (5.46)$$

*is the center of mass momentum operator.*

The proof of Theorem 2 is given in Sec. B.2 of Appendix B.

### 5.3.3 Condition on the potential

We discuss the functional equation (5.37) which determines the interaction potential. It appears naturally in the context of the following question: Find a two-body interaction potential for which the ground state wave function to the eigenvalue zero factorizes into a product of two-body terms. This problem was first studied by Sutherland [8]. To be more precisely we consider the Schrödinger equation with the corresponding Ansatz

$$\left( -\sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} + \sum_{j \neq l}^{N} V(x_j - x_l) \right) \Psi = 0 \quad , \quad \Psi \propto \prod_{j<l}^{N} \exp(-F(x_j - x_l)) \ . \tag{5.47}$$

Taking the second derivative of the wave function yields

$$\sum_{n=1}^{N} \frac{\partial^2 \Psi}{\partial x_n^2} = \left( \sum_{n \neq j}^{N} \left[ f^2(x_n - x_j) - f'(x_n - x_j) \right] + \sum_{n \neq j \neq l}^{N} f(x_n - x_j) f(x_n - x_l) \right) \Psi \ , \tag{5.48}$$

where we use the notation $f \equiv F'$. The terms in the square brackets on the right hand side only depend on the difference of two particle positions. They resemble the relation (5.43) and can be identified as two-body interaction potential. However, the remaining terms mix three particle positions. In order for the Ansatz in Eq. (5.47) to work, these terms are required to factorize into two-body terms as well. This yields a condition on $f$ in form of the functional equation

$$f(x)f(y) + f(x)f(y) + f(y)f(z) = W(x) + W(y) + W(z) \quad , \quad x + y + z = 0 \ , \tag{5.49}$$

where $W(x)$ is an arbitrary function. The equation above has been derived by Sutherland [8]. Its general solution was found by Calogero [99, 100].

The condition (5.37) corresponds to the special case of Eq. (5.49) where the right hand side is constant i.e. $3W \equiv$ const. This is the same condition as it was found for identical Fermions. A detailed discussion of the solution can be found in Ref. [67]. Adapting the results there, the solution to the functional equation (5.37) can be written as

$$f(x) = z\lambda \coth(zx - \kappa) \ . \tag{5.50}$$

Here $z = a + \imath b$ with $a, b \in \mathbb{R}$ and $\lambda$,$\kappa \in \mathbb{C}$. The condition of real potential restricts $z$ to be either real or purely imaginary and $\lambda$ to be real. However, it turns out that in the present case $\lambda$ trivially rescales the three parameters $\lambda_1$, $\lambda_2$ and $\lambda_3$ in Eq. (5.38). Therefore we set $\lambda = 1$. This has been anticipated in Theorem 2 [2]. Since the solution for the functional equation is the same as for identical Fermions, the treatable interaction potentials coincide as well. Adapting these results, we state in Table 5.1 a list of potentials $V(x)$ together with the corresponding functions $f(x), F(x)$ and the constants $a, b$ and $\kappa$ which can be derived from Eq. (5.50).

| Type | $a$ | $b$ | $\kappa$ | $f(x)$ | $V(x)$ | $F(x)$ |
|---|---|---|---|---|---|---|
| (I) | 0 | free | 0 | $b\cot(bx)$ | $\frac{b^2}{\sin^2(xb)}$ | $\ln\lvert\sin(bx)\rvert$ |
| (II) | 0 | 0 | 0 | $1/x$ | $\frac{1}{x^2}$ | $\ln\lvert x\rvert$ |
| (III) | free | 0 | 0 | $a\coth(ax)$ | $\frac{a^2}{\sinh^2(xa)}$ | $\ln\lvert\sinh(ax)\rvert$ |
| (IV) | free | 0 | $\imath\pi/2$ | $a\tanh(ax)$ | $-\frac{a^2}{\cosh^2(xa)}$ | $\ln\lvert\cosh(ax)\rvert$ |

Table 5.1: A list of interaction potentials $V(x)$ and the corresponding functions $f(x)$ and $F(x)$ which can be derived from the solution (5.50) of the functional equation.

The parameters $a$ and $b$ have the dimension of a inverse length. For the potentials (I) and (II) $a = 0$. The potential (I) is periodic with period $2\pi/b$ and the corresponding system is referred to as trigonometric Calogero-Moser-Sutherland (CMS) model. For this potential only periodic boundary conditions are allowed. Taking the limit $b \to 0$ of potential (I) leads to the inverse square potential (type (II)). Here only scattering boundary conditions are allowed. The system is also known as rational CMS model. For the potentials (III) and (IV) $b = 0$ and $a$ is arbitrary. Systems with a pairwise interaction potential of type (III) are called hyperbolic CMS models, whereas the potential (IV) corresponds to the Morse potential. Both models describe scattering systems. Also the $\delta$-potential can be obtained. This is seen by using the representation

$$\delta(x) = \lim_{a\to\infty} \frac{1}{2}\frac{a}{\cosh^2(ax)} \tag{5.51}$$

of the $\delta$-function. Multiplying the potential (IV) with the coupling constant $g$ and using the representation above, the $\delta$-potential is obtained by

$$\lim_{\substack{a\to\infty\\ g\to 0}} -g\frac{a^2}{\cosh^2(ax)} = -2c\delta(x)\ , \tag{5.52}$$

where the product $ag = c$ determines the interaction strength of the potential. As we will see below, creation and annihilation operators for the $\delta$-potential with finite interaction strength can only be constructed for $m_1 = m_2$.

### 5.3.4 Condition on the coupling constants and masses

The models discussed above are known to be exactly solvable already for a long time. The main statement of Theorem 2 concerns the exact solvability of the above mentioned

[2]For $\lambda \neq 1$ the third condition in Eq. (5.38) is $(m_1 - m_2)(\lambda_2 - \lambda^{-1}) = 0$ and the coupling constants in (5.42) are $g_j = \lambda_j(\lambda_j + \lambda^{-1})/(2m_j)$, $j = 1, 2$ and $g_3 = \lambda_3(\lambda_3 + \lambda^{-1})(m_1^{-1} + m_2^{-1})/2$.

systems beyond the limitation of equal masses and coupling constants. Here we discuss the necessary conditions (5.38) and (5.42).

We start with the case of equal masses. For $m_1 = m_2$ the third condition in Eq. (5.38) is fulfilled for arbitrary $\lambda_3$. Combining the first two conditions in Eq. (5.38) with Eq. (5.42) yields for the parameters $\lambda_1,\ \lambda_2,\ \lambda_3$ and the coupling constants the relations

$$\lambda_1 = \lambda_2 = \lambda_3 \qquad , \qquad g_1 = g_2 = \frac{g_3}{2} = \frac{\lambda_1(\lambda_1+1)}{2m_1} \ . \tag{5.53}$$

In particular the limit $g_j \to 0$ or equivalently $\lambda_1 \to 0$ can be taken. Hence the $\delta$-potential with finite interaction strength can be obtained from the potential of type (IV) via the relation (5.52).

Using the notation $z_1 = x_1, \ldots, z_N = x_N, z_{N+1} = y_1, \ldots, z_{N+M} = y_M$, the Hamiltonian (5.16) acquires the form

$$H_{N,M}(\mathbf{z}) \ = \ \frac{1}{2m_1}\left( -\sum_{n=1}^{N+M} \frac{\partial^2}{\partial z_n^2} + \lambda_1(\lambda_1+1) \sum_{i\neq j}^{N+M} V(z_i - z_j) \right) \ . \tag{5.54}$$

The mass $m_1$ sets the energy scale such that effectively the only free parameter is $\lambda_1$ which determines the coupling constant. The Hamiltonian has the same form as for identical particles. The difference of the creation and annihilation functions for those and for spin one-half Fermions are the statistical functions (5.24)-(5.27). In the present context the case of identical Fermions is restored by setting the number of particles for one species to zero i.e. either $M = 0$ or $N = 0$.

We consider the case $m_1 \neq m_2$. The third condition in Eq. (5.38) now requires $\lambda_3 = 1$. Combining the first two conditions yields the following relations between the masses and $\lambda_1$ and $\lambda_2$

$$\lambda_1 = \frac{m_1}{m_2} \qquad , \qquad \lambda_1 = \frac{1}{\lambda_2} \ . \tag{5.55}$$

Together with $\lambda_3 = 1$ these are three conditions for the parameters in the Hamiltonian (5.16). Thus only two of the originally five parameters in $H_{N,M}$ are independent.

Using the relations (5.55), the coupling constants (5.43) can be expressed as functions of $\lambda_1$ and $m_1$ only

$$g_1 = \frac{\lambda_1(\lambda_1+1)}{2m_1} \quad , \quad g_2 = \frac{1}{2m_1}\left(1 + \frac{1}{\lambda_1}\right) \quad , \quad g_3 = \frac{1}{m_1}(1+\lambda_1) \ . \tag{5.56}$$

Substituted into the Hamiltonian (5.16) this yields

$$H_{N,M}(\mathbf{z}) \ = \ \frac{1}{2m_1}\left( -\sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} - \lambda_1 \sum_{m=1}^{M} \frac{\partial^2}{\partial y_m^2} + \lambda_1(\lambda_1+1) \sum_{i\neq j}^{N} V(x_i - x_j) \right. \tag{5.57}$$

$$+ \quad \left(1+\frac{1}{\lambda_1}\right)\sum_{i\neq j}^{M} V(y_i-y_j) + 2(1+\lambda_1)\sum_{i=1}^{N}\sum_{j=1}^{M} V(x_i-y_j)\Bigg) \ .$$

Again the energy scale is set by $m_1$ such that the right hand site of Eq. (5.57) effectively depends only on the parameter $\lambda_1$, which according to Eq. (5.55) is determined by the ratio of masses. For unequal masses but finite values of the mass ratio $m_1/m_2$ also the coupling constants for the potentials (I)-(IV) remain finite. For the $\delta$-potential the situation is different. Since a local potential is invisible for identical Fermions, the first two interaction terms in Eq. (5.57) can be dropped in this case. However, due to the specific form of the coupling constant $g_3$, the limit $g_3 \to 0$ can not be taken for different and finite masses. Therefore an exact treatment of $\delta$-potential with finite interaction and different particle masses is not available within the present approach.

The case $\lambda_1 \ll 1$ corresponds to the situation, where the spin-up particles are much heavier than the spin-down particles. Following Eq. (5.56), the coupling constant $g_2$ dominates followed by $g_3$ and $g_1$. On the other hand, if $\lambda_1 \gg 1$, the spin-down particles are heavy in comparison to the spin-up particles and the hierarchy of the coupling constants is the other way around i.e. $g_1 < g_3 < g_2$. The non-interacting case for $m_1 \neq m_2$ is obtained by setting $F = f = V \equiv 0$.

## 5.4 Applications

According to Eqs. (5.18) and (5.19) the wave function with the number of spin-down particles increased or decreased by one is obtained form the $N+M$ particle wave function by integration with a corresponding integration kernel. The multiple integrals however, are by far to complicated in order to allow for a general evaluation. Nevertheless it is instructive to see how the mechanism of the creation and annihilation operators works for the simplest cases.

### 5.4.1 Free particles

We consider the case of non-interacting spin one-half Fermions. Our starting point is the corresponding wave function for $N$ spin-down and $M$ spin-up Fermions. In coordinate representation the wave function is the product of two Slater determinants. It reads

$$\Psi_{N,M}(\mathbf{z},\mathbf{k},\mathbf{q}) \propto \det\left[e^{2\imath m_1 k_j x_n}\right]_{j,n=1,\ldots,N} \det\left[e^{2\imath m_2 q_l y_m}\right]_{l,m=1,\ldots,M} , \tag{5.58}$$

where $\mathbf{k} = \{k_j\}_{j=1,\ldots,N}$ and $\mathbf{q} = \{q_l\}_{l=1,\ldots,M}$ denote the quasi-momenta of the spin-down and the spin-up particles, respectively. For the non-interacting case, we set $F \equiv 0$ in Eqs. (5.39) and (5.40).

We consider the action of the creation operator. According to Eqs. (5.24), (5.26) and (5.39), the full creation function reads

$$\bar{a}_k^\dagger(\mathbf{z}',\mathbf{z}) = \frac{2^{-N}}{(N+1)!} \det\Big[\mathrm{sgn}(x'_n - x_l)\Big|1\Big]_{\substack{n=1,\ldots,N+1\\ l=1,\ldots,N}} \exp\left(2\imath k m_1\left(\sum_{n=1}^{N+1} x'_n - \sum_{m=1}^{N} x_m\right)\right)$$

$$\times \quad \frac{1}{2^M M!} \det\left[\mathrm{sgn}(y'_m - y_l)\right]_{m,l=1,\ldots,M} \exp\left(2\imath k m_2 \sum_{m=1}^{M} (y'_m - y_m)\right) \ . \tag{5.59}$$

Substituting Eqs. (5.58) and (5.59) into the recursion relation (5.18), we find

$$\begin{aligned} \Psi_{N+1,M}(\mathbf{z}', \mathbf{k}', \mathbf{q}) \quad \propto \quad & \exp\left(2\imath m_1 k \sum_{n=1}^{N+1} x'_n + 2\imath m_2 k \sum_{m=1}^{M} y'_m\right) \\ & \int_{\Omega^N} d[\mathbf{x}] \det\left[\mathrm{sgn}(x'_n - x_l)\Big|1\right]_{\substack{n=1,\ldots,N+1 \\ l=1,\ldots,N}} \det\left[e^{2\imath m_1 (k_j - k) x_n}\right]_{j,n=1,\ldots,N} \\ & \int_{\Omega^M} d[\mathbf{y}] \det\left[\mathrm{sgn}(y_m - y'_l)\right]_{m,l=1,\ldots,M} \det\left[e^{2\imath m_2 (q_l - k) y_m}\right]_{l,m=1,\ldots,M} \ . \end{aligned} \tag{5.60}$$

We expand the statistical function in the second line with respect to the last column and use the linearity of the determinant as well as relation (3.12). This yields

$$\begin{aligned} \Psi_{N+1,M}(\mathbf{z}', \mathbf{k}', \mathbf{q}) \quad \propto \quad & \sum_{j=1}^{N+1} (-1)^{j+N+1} \det\left[\int_{\Omega} dx \ \mathrm{sgn}(x'_n - x) e^{2\imath m_1 (k_i - k) x}\right]_{\substack{n=1,\ldots,N+1 \neq j \\ i=1,\ldots,N}} \\ & \exp\left(2\imath m_1 k \sum_{n=1}^{N+1} x'_n + 2\imath m_2 k \sum_{m=1}^{M} y'_m\right) \det\left[\int_{\Omega} dy \ \mathrm{sgn}(y'_m - y) e^{2\imath m_2 (q_l - k) y}\right]_{m,l=1,\ldots,M} \ . \end{aligned} \tag{5.61}$$

For open boundary conditions we have $\Omega = \mathbb{R}$. The integrals in the equation above can be written as

$$-\int_{-\infty}^{x'_n} dx \ e^{2\imath m_1 (k_i - k) x} + \int_{x'_n}^{+\infty} dx \ e^{2\imath m_1 (k_i - k) x} = -\frac{e^{2\imath m_1 (k_i - k) x'_n}}{\imath m_1 (k_i - k)} \ , \tag{5.62}$$

where the first (second) term on the left hand side has been regularized by slightly shifting $k \to k \pm \imath\epsilon$ into the upper (lower) complex plane, doing the integral and afterwards taking the limit $\epsilon \to 0$. With Eq. (5.62) we obtain

$$\Psi_{N+1,M}(\mathbf{z}', \mathbf{k}', \mathbf{q}) \propto \det\left[e^{2\imath m_1 k_j x'_n}\Big| e^{2\imath m_1 k x'_n}\right]_{\substack{n=1,\ldots,N+1 \\ j=1,\ldots,N}} \det\left[e^{2\imath m_2 q_l y'_m}\right]_{l,m=1,\ldots,M} \ , \tag{5.63}$$

which is the wave function of free spin one-half Fermions with the number of spin-down particles increased by one. Thus, the action of the creation operator appends a column and a row to the corresponding Slater determinate.

We turn to the action of the annihilation operator. Substituting Eqs. (5.25), (5.27) and (5.40) into the recursion relation for the annihilation operator (5.19), we have

$$\Psi_{N-1,M}(\mathbf{z}'', \mathbf{k}'', \mathbf{q}) \quad \propto \quad \exp\left(2\imath m_1 k \sum_{n=1}^{N-1} x''_n + 2\imath m_2 k \sum_{m=1}^{M} y'_m\right) \tag{5.64}$$

$$\int\limits_{\Omega^N} d[\mathbf{x}] \det\left[\mathrm{sgn}(x_l - x_n'')\Big|1\right]_{\substack{n=1,\ldots,N-1\\ l=1,\ldots,N}} \det\left[e^{2\imath m_1(k_j-k)x_n}\right]_{j,n=1,\ldots,N}$$
$$\int\limits_{\Omega^M} d[\mathbf{y}] \det\left[\mathrm{sgn}(y_m - y_l'')\right]_{m,l=1,\ldots,M} \det\left[e^{2\imath m_2(q_l-k)y_m}\right]_{l,m=1,\ldots,M} ,$$

The evaluation of this expression follows along the same scheme as for the creation operator. The outcome is

$$\Psi_{N-1,M}(\mathbf{z}'', \mathbf{k}'', \mathbf{q}) \propto \det\left[\frac{e^{2\imath m_1 k_j x_n''}}{\imath(k_j - k)}\Big|\delta(k_j - k)\right]_{\substack{n=1,\ldots,N-1\\ j=1,\ldots,N}} \det\left[e^{2\imath m_2 q_l y_m''}\right]_{l,m=1,\ldots,M} \quad (5.65)$$

A comparison with the original wave function (5.58) reveals that the last column of the Slater determinant for the spin-down particles has been replaced by $\delta$-functions. This indicates that the action of the annihilation operator yields a non-zero result only if a spin-down particle with momentum $k$ is present. From Eq. (5.65) follows

$$\langle \mathbf{z}''|\hat{a}_k|\Psi_{N,M}(\mathbf{k})\rangle \propto \begin{cases} 0 & \text{for} \quad k \notin \mathbf{k} \\ \Psi_{N-1,M}(\mathbf{z}'', \mathbf{k}'', \mathbf{q}) & \text{for} \quad k \in \mathbf{k} \end{cases} , \quad (5.66)$$

where $\mathbf{k}'' = \mathbf{k}/k$. This is the expected result for the annihilation operator.

For periodic boundary conditions we have $\Omega = [0, L]$. Since we require the old and the new wave function to be periodic with period $L$, all quasi-momenta need to be multiple integers of $2\pi/L$ i.e. $\mathbf{k} = 2\pi\mathbf{n}/L$ and $\mathbf{q} = 2\pi\mathbf{m}/L$, where $\mathbf{n}$ and $\mathbf{m}$ are two sets of unequal integers. The integrals are finite such that no regularization is necessary. Apart from that, the evaluation for the action of the creation operator does not change at all and the outcome is again given by Eq. (5.63). The result for the action of the annihilation operator is analogously. What changes is that the $\delta$-functions in Eq. (5.64) are replaced by Kronecker-$\delta$'s i.e. $2\pi\delta(k - k') \to L\delta_{k,k'}$.

### 5.4.2 Particles with contact interaction

We consider particles with repulsive $\delta$-function potential. For finite interaction strength, an exact solution is available only for equal masses. Therefore we set throughout the following $m_1 = m_2 = 1/2$. The exact wave functions can be constructed by means of the Bethe-Ansatz and have been discussed extensively in the first part. However, their evaluation in the present context for an arbitrary number of spin-up and spin-down particles seems to be a cumbersome task. To nevertheless see the applicability of the approach, we consider the simplest case. Our starting point is the wave function for a spin-up particle with momentum $k_1$

$$\Psi_{0,1}(y, k_1) \propto e^{\imath k_1 y} . \quad (5.67)$$

Upon this we wish to create a spin-down particle with momentum $k_2$ that interacts with the spin-up particle via repulsive $\delta$-potential. We consider the scattering system

and hence $\Omega = \mathbb{R}$. According to the recursion relation (5.18) the corresponding wave function can be obtained by

$$\Psi_{1,1}(\mathbf{z}', \mathbf{k}') = \int_{-\infty}^{+\infty} dy I_{1\uparrow}(y', y) a^\dagger_{k_2}(x', y', y) \Psi_{0,1}(y, k_1) \ , \tag{5.68}$$

where the statistical function reads $I_{1\uparrow}(y', y) = \mathrm{sgn}(y' - y)/2$ and the creation function is given by

$$a^\dagger_{k_2}(x', y', y) = \exp\Big(\imath(k_1 + k_2)(x' + y') - \imath k_2 y + \Big(c|y' - y| - c|x' - y'| + c|x' - y|\Big) \tag{5.69}$$

Substitution into Eq. (5.68) yields

$$\begin{aligned} \Psi_{1,1}(\mathbf{z}', \mathbf{k}') \quad \propto \quad e^{\imath(k_1+k_2)(x'+y') - c|x'-y'|} \Bigg( & e^{cy'} \int_{-\infty}^{y'} dy \ e^{(\imath k_{12} - c)y + c|x'-y|} \\ & - e^{-cy'} \int_{y'}^{+\infty} dy e^{(\imath k_{12} + c)y + c|x'-y|} \Bigg) \ . \end{aligned} \tag{5.70}$$

Here we use the notation $k_{12} = k_1 - k_2$. To evaluate this expression further, we consider the sector where the primed variables are ordered such that $x' > y'$. Then the absolute values in the exponent can be eliminated. From Eq. (5.70) we obtain

$$\Psi_{1,1}(\mathbf{z}', \mathbf{k}')\Big|_{x'>y'} \propto \frac{e^{\imath k_1 y' + \imath k_2 x'}}{\imath k_{12} - 2c} + \frac{e^{\imath k_2 y' + \imath k_1 x'}}{\imath k_{12} + 2c} - \frac{e^{\imath k_1 x' + \imath k_2 y'} - e^{\imath k_2 x' + \imath k_1 y'}}{\imath k_{12}} \ , \tag{5.71}$$

where we have used that $c < 0$. This ensures the convergence of the integrals in Eq. (5.70). For the reversed ordering of particles i.e. $y' > x'$, one finds the same expression but with $c \to -c$, that is

$$\Psi_{1,1}(\mathbf{z}', \mathbf{k}')\Big|_{y'>x'} \propto \frac{e^{\imath k_1 y' + \imath k_2 x'}}{\imath k_{12} + 2c} + \frac{e^{\imath k_2 y' + \imath k_1 x'}}{\imath k_{12} - 2c} - \frac{e^{\imath k_1 x' + \imath k_2 y'} - e^{\imath k_2 x' + \imath k_1 y'}}{\imath k_{12}} \ . \tag{5.72}$$

Combining Eqs. (5.71) and (5.72) the wave function for an arbitrary ordering of particles can be cast into the form

$$\begin{aligned} \Psi_{1,1}(\mathbf{z}', \mathbf{k}') \propto \frac{1}{2\imath k_{12}} \det\left[e^{\imath k_j x'} | e^{\imath k_j y'}\right] \\ + \frac{1}{k_{12}^2 + 4c^2} \left[ \left( \frac{\imath k_{12}}{2} + c\,\mathrm{sgn}(x' - y') \right) e^{\imath k_1 y' + \imath k_2 x'} + \left( \frac{\imath k_{12}}{2} - c\,\mathrm{sgn}(x' - y') \right) e^{\imath k_1 x' + \imath k_2 y'} \right] \ . \end{aligned} \tag{5.73}$$

The first term on the right hand side is antisymmetric. It corresponds to the fermionic result. The terms in the second line are the wave function for two Bosons with $\delta$-interaction. This term is symmetric. The superposition of the two contributions has

no well defined symmetry. This reflects the fact that both particles are distinguishable. The wave function (5.73) is easily shown to be an eigenfunction of the Hamiltonian

$$H_{1,1}(\mathbf{z}') = -\frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial y'^2} - 2c\delta(x' - y') \tag{5.74}$$

to the eigenvalue $k_1^2 + k_2^2$.

A generalization of the direct calculation presented above to an arbitrary number of particles seems not to be in reach. The main obstacle is the fast growing number of orderings for higher particle numbers. As the explicit calculation above showed, we needed to consider the cases $x' > y'$ and $y' > x'$ separately. These are the only two orderings which are possible for two particles. However, for $N$ particles there are $N!$ possible orderings of the primed variables. Therefore carrying out the integrals as in the explicit manner above becomes forbiddingly complicated.

## 5.5 Summary of Part II

We studied the exact solvability of one-dimensional systems with two sorts of Fermions. For certain types of interactions, such as the inverse square potential or its periodic version, the trigonometric CMS-model, such models are known to be exactly solvable for equal masses. Results concerning the exact solvability beyond this limitation are available only for some special cases [95, 96, 101, 98].

Starting from a general setup, we studied the case where the masses of the two particle species as well as the coupling constants are distinct. The corresponding system is described in first and second quantized form by the Hamiltonians (5.16) and (5.7), respectively. Our main result, stated in Theorem 2, consists in the exact construction of particle creation and annihilation operators for such models in coordinate space. The potentials for which this is possible are the same as for identical Fermions. However, in addition to that we found the necessary relations between the masses and the coupling constants for the applicability of the approach to a mixture of two particle species with different masses. It turned out that in this case the Hamiltonian depends on only a single parameter which is the ratio of masses $m_1/m_2$ (see Eq. (5.57)).

By means of the creation operator, the eigenfunctions can be constructed recursively. The resulting expressions for the many-body wave function however, are highly involved and do not seem to allow for a further evaluation. Nevertheless, from a conceptual point of view our result is important, since it proofs two component systems for various types of interaction potentials to be exactly solvable beyond the limitation of equal masses.

# Chapter 6

# Conclusion and Outlook

The first part of this thesis was devoted to the Yang-Gaudin model of spin one-half Fermions with repulsive contact interaction. We gave a reformulation of the exact eigenfunctions which turns out to be particular well suited for the highly imbalanced case. In that manner, the exact derivation of several thermodynamical expressions for the case that one or two minority Fermions move in a Fermi-sea of majority particles became feasible. We calculated expectation values, density-density correlation functions and the single particle Green's function. It turns out that the free momenta of the minority particles crucially affect the properties of all of these quantities. In particular in the hardcore limit this became evident: If only a single minority particle is present, its Green's function shows a transition from the one of impenetrable Bosons to that of free Fermions in dependence of the extra particle's momentum. Moreover, a perturbative evaluation of the survival probability showed that the interaction causes a decay of the non-interacting initial state only if the minority particle's momentum is located outside the Fermi-sea. On the one hand, this indicates a substantial difference between the cases that the momentum is inside or outside the Fermi-sea. On the other hand, it shows the limited applicability of perturbation theory in the present case and underlines the importance of the non-perturbative approach to evaluate the above mentioned static quantities. A non-perturbative evaluation of dynamical quantities is a challenging as well as a demanding task for the future.

In the second part, we studied to what extent mixtures of two particle species with different masses are exactly solvable. The approach we used to address that question is rooted on the explicit construction of particle creation and annihilation operators and has previously been developed for identical Fermions. Whereas the potentials for which the method applies are the same as in that case, it turns out that a exact solution for different masses is available only if certain relations among the coupling constants and the masses hold. This results in a Hamiltonian with the ratio of masses as the only free parameter. The coupling constants are functions of this parameter. Whether or not this result can be generalized to mixtures with more than two particle species is an open question.

# Appendix A

# Derivations for Part I

## A.1 Proof of Theorem 1

We proof Theorem 1. For $M = 1$ this might be done by directly acting with the Hamiltonian (2.2) onto the wave function (2.11). However, for an arbitrary number of spin-up particles this becomes cumbersome. A more convenient way to prove the first part of Theorem 1 is to show that the wave function fulfills the following boundary conditions:

1. $\Psi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda})$ fulfills the free Schrödinger equation in every sector.
2. $\Psi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda})$ is continuous everywhere.
3. Let $y_m$ and $x_n$ be adjacent. Then the first derivative of $\Psi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda})$ evaluated at $x_n = y_m$ is discontinuous such that

$$\left(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial y_m}\right) \Psi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda})\Big|_{x_n - y_m = 0^-}^{x_n - x_m = 0^+} = 4c\Psi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda})\Big|_{x_n = y_m}, \quad \text{(A.1)}$$

   where $0^{\pm}$ means that zero is approached from above/below.

The first condition follows directly from the locality of the interaction potential. The second and the third condition arise, since the potential is a $\delta$-function. In particular the third condition is obtained by integrating the stationary Schrödinger equation (2.1) from $x_n - y_m = -\epsilon$ to $x_n - y_m = +\epsilon$, where $\epsilon > 0$ and taking afterwards the limit $\epsilon$ to zero.

As can be seen right from its definition, the wave function (2.11) is continuous and fulfills the free Schrödinger equation within each sector. Also the continuity of $\Psi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda})$ at the boundaries where $x_n = x_j$ is obvious. To show the continuity of Eq. (2.11) at the positions where $x_n = y_m$ we use

$$\Psi(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{\Lambda})\Big|_{x_n - y_m = 0^{\pm}} \propto \sum_{R \in S(M)} \mathrm{sgn}(R) \prod_{j<l}^{M} [\imath(\Lambda_{Rj} - \Lambda_{Rl}) + 2c\,\mathrm{sgn}(y_l - y_j)] \quad \text{(A.2)}$$

$$\det\left[\ldots\Big|\,[\imath(k_j-\Lambda_{Rm})\pm c]\prod_{s\neq m}^{M}A_j(\Lambda_{Rs},y_m-y_s)e^{\imath k_j y_m}\Big|\ldots\right].$$

The dots indicate that all other columns of the determinant (2.12) remain unchanged. We use the linearity of the determinant. Then taking the difference, the $n$-th and the $N+m$-th column become linearly depended and consequently the determinant vanishes. This proves the continuity of $\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})$ at $x_n=y_m$. In order to show the continuity of the wave function at the borders where $y_\mu=y_\nu$ we use

$$\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})\Big|_{y_\mu-y_\nu=0^\pm}\propto\sum_{R\in S(M)}\operatorname{sgn}(R)\left[(\Lambda_{R\mu}-\Lambda_{R\nu})^2+4c^2\right] \tag{A.3}$$

$$\prod_{\substack{j<l\\ j\neq\mu\\ l\neq,\nu}}^{M}[\imath(\Lambda_{Rj}-\Lambda_{Rl})+2c\operatorname{sgn}(y_l-y_j)]\Big|_{y_\mu-y_\nu=0^\pm}\det\left[\ldots\Big|\prod_{s\neq\mu,\nu}^{M}A_j(\Lambda_{Rs},y_\mu-y_s)e^{\imath k_j y_\mu}\Big|\right.$$

$$\left.\ldots\Big|[\imath(k_j-\Lambda_{R\mu})\mp c]\prod_{s\neq\mu,\nu}^{M}A_j(\Lambda_{Rs},y_\mu-y_s)e^{\imath k_j y_\mu}\Big|\ldots\right].$$

Taking the difference and employing the linearity of the determinant, it is again seen that the $\mu$-th and the $\nu$-th column become linearly dependent when the difference is taken and hence the determinant vanishes. This completes the proof regarding the continuity of $\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})$.

We consider the condition (A.1). The derivative of $\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})$ with respect to $x_n$ yields

$$\frac{\partial\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})}{\partial x_n}\Big|_{x_n-y_m=0^\pm}\propto\sum_{R\in S(M)}\operatorname{sgn}(R)\prod_{j<l}^{M}[\imath(\Lambda_{Rj}-\Lambda_{Rl})+2c\operatorname{sgn}(y_l-y_j)] \tag{A.4}$$

$$\det\left[\ldots|\imath k_j\,[\imath(k_j-\Lambda_{Rm})\pm c]\prod_{i\neq m}^{M}A_j(\Lambda_{Ri},y_m-y_i)e^{\imath k_j y_m}\Big|\ldots\right].$$

Taking the difference we have

$$\frac{\partial\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})}{\partial x_n}\Big|_{x_n-y_m=0^-}^{x_n-y_m=0^+}\propto\sum_{R\in S(M)}\operatorname{sgn}(R)\prod_{j<l}^{M}[\imath(\Lambda_{Rj}-\Lambda_{Rl})+2c\operatorname{sgn}(y_l-y_j)] \tag{A.5}$$

$$\det\left[\ldots|2c\imath k_j\prod_{i\neq m}^{M}A_j(\Lambda_{Ri},y_m-y_i)e^{\imath k_j y_m}\Big|\ldots\right].$$

For the derivative of $\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})$ with respect to $y_m$ we find

$$\frac{\partial\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})}{\partial y_m}\Big|_{x_n-y_m=0^-}^{x_n-y_m=0^+}\propto \tag{A.6}$$

$$\sum_{\substack{R\in\\ S(M)}} \mathrm{sgn}(R) \left[ \left( \frac{\partial}{\partial y_m} \prod_{j<l}^{M} [\imath(\Lambda_{Rj} - \Lambda_{Rl}) + 2c\mathrm{sgn}(y_l - y_j)] \right) \Phi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda}) \Big|_{x_n-y_m=0^-}^{x_n-y_m=0^+} \right. \tag{A.7}$$

$$\left. \prod_{j<l}^{M} [\imath(\Lambda_{Rj} - \Lambda_{Rl}) + 2c\mathrm{sgn}(y_l - y_j)] \frac{\partial\Phi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})}{\partial y_m} \Big|_{x_n-y_m=0^-}^{x_n-y_m=0^+} \right] . \tag{A.8}$$

Doing the derivative in line (A.7) yields a factor $\delta(y_n - y_m),\ n = 1,\ldots,M \neq m$. When multiplied with $\Phi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})$ and evaluated at $x_n - y_m = 0^{\pm}$ the difference vanishes, since $\Phi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})$ is continuous at $x_n = y_m$. Thus we have

$$\frac{\partial\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})}{\partial y_m}\Big|_{x_n-y_m=0^-}^{x_n-y_m=0^+} = \frac{\partial\Phi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})}{\partial y_m}\Big|_{x_n-y_m=0^-}^{x_n-y_m=0^+} . \tag{A.9}$$

According to Eq. (2.12) every column of the determinant $\Phi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})$ depends on $y_m$. To further evaluate its derivate with respect to $y_m$ we use

$$\frac{\partial}{\partial y_m} \prod_{s=1}^{M} \left[\imath(k_j - \Lambda_{R(s)}) + 2c\mathrm{sgn}(x_l - y_s)\right] e^{\imath k_j x_l}\Big|_{x_n-y_m=0^{\pm}} \tag{A.10}$$

$$= \begin{cases} 0 & \text{for} \quad l \neq n \\ -2c\delta(x_n - y_m) \prod\limits_{s\neq m}^{M} A_j(\Lambda_{Rs}, y_m - y_s) e^{\imath k_j y_m} & \text{for} \quad l = n \end{cases} .$$

By means of the equation above it is then easily seen that the derivative with respect to the first $N$ columns of $\Phi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})$ vanishes. Carrying out the derivatives with respect to the last $M$ columns we obtain

$$\frac{\partial\Phi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})}{\partial y_m}\Big|_{x_n-y_m=0^{\pm}} \propto \tag{A.11}$$

$$4c \sum_{l\neq m}^{M} \delta(y_m - y_l) \det \left[ \ldots \Big| \prod_{\substack{s=1\\ \neq m,l}}^{M} A_j(\Lambda_{Rs}, y_l - y_s) e^{\imath k_j y_l} \Big| \ldots \right] +$$

$$\det \left[ \ldots \Big| \imath k_j \prod_{s\neq m}^{M} A_j(\Lambda_{Rs}, y_m - y_s) e^{\imath k_j y_m} \Big| \ \ldots \Big| \mp c \prod_{s\neq m}^{M} A_j(\Lambda_{Rs}, y_m - y_s) e^{\imath k_j y_m} \Big| \ldots \right] .$$

When the difference is taken, the terms in the second line of Eq. (A.11) drop out and together with Eq. (A.9) we have

$$\frac{\partial\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})}{\partial y_m}\Big|_{x_n-y_m=0^-}^{x_n-y_m=0^+} = \sum_{R\in S(M)} \mathrm{sgn}(R) \prod_{j<l}^{M} [\imath(\Lambda_{Rj} - \Lambda_{Rl}) + 2c\mathrm{sgn}(y_l - y_j)] \tag{A.12}$$

$$\det \left[ \ldots \Big| \imath k_j \prod_{s\neq m}^{M} A_j(\Lambda_{Rs}, y_m - y_s) e^{\imath k_j y_m} \Big| .. \Big| - 2c \prod_{s\neq m}^{M} A_j(\Lambda_{Rs}, y_m - y_s) e^{\imath k_j y_m} \Big| \ldots \right] .$$

Observing that also

$$\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})\Big|_{x_n=y_m} \propto \sum_{R\in S(M)} \mathrm{sgn}(R) \prod_{j<l}^{M} [\imath(\Lambda_{Rj}-\Lambda_{Rl}) + 2c\mathrm{sgn}(y_l-y_j)] \qquad \text{(A.13)}$$
$$\det\left[\ldots\left|\imath k_j \prod_{i\neq m}^{M} A_j(\Lambda_{Ri}, y_m - y_i) e^{\imath k_j y_m}\right|\ldots\right]$$

leads in combination with Eqs. (A.5) and (A.12) to Eq. (A.1). This completes the proof that the wave function in Eq. (2.11) is an eigenfunction to the Hamiltonian (2.2). The corresponding eigenvalue is given by $E$, see Eq. (2.15). This follows directly from the fact that $\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})$ fulfills the free Schrödinger equation in each sector. The latter property implies furthermore that $\Psi(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{\Lambda})$ is also an eigenfunction of the center of mass momentum operator to the eigenvalue $K$ given in Eq. (2.15). This completes the proof of Theorem 1.

## A.2 Eigenfunctions for the Hubbard model

We proof the expression (2.68) to be an eigenfunction of the Hamiltonian (2.62). While doing so, we use the notation

$$\bar{A}_j(x) = \imath\left(\frac{\sin(k_j a)}{a} - \Lambda\right) + c\mathrm{sgn}(x)\ . \qquad \text{(A.14)}$$

Starting with the kinetic term for $x_l$ we consider

$$\begin{aligned}
&\sum_{s=\pm 1} \varphi(x_1,\ldots,x_l+sa,\ldots,x_N,y) \qquad \text{(A.15)}\\
=\ & \det\left[\cdots\left|\left(\bar{A}_j(x_l+a-y)e^{+\imath k_j a} + \bar{A}_j(x_l-a-y)e^{-\imath k_j a}\right)e^{\imath k_j x_l}\right|\cdots\left|e^{\imath k_j y}\right.\right]\\
=\ & \det\left[\cdots|2\cos(k_j a)\bar{A}_j(x_l-y)e^{k_j x_l}\right|\cdots|e^{\imath k_j y}\Big]\\
+\ & \det\left[\cdots\left|c(\mathrm{sgn}(x_l+a-y)-\mathrm{sgn}(x_l-y))e^{\imath k_j(a+x_l)}\right|\cdots\left|e^{\imath k_j y}\right.\right]\\
+\ & \det\left[\cdots\left|c(\mathrm{sgn}(x_l-a-y)-\mathrm{sgn}(x_l-y))e^{\imath k_j(-a+x_l)}\right|\cdots\left|e^{\imath k_j y}\right.\right].
\end{aligned}$$

By means of the relations

$$\mathrm{sgn}(x+a-y)-\mathrm{sgn}(x-y) = \delta_{x-y,0}+\delta_{x-y,-a}\ , \qquad \text{(A.16)}$$
$$\mathrm{sgn}(x-a-y)-\mathrm{sgn}(x-y) = -\delta_{x-y,0}-\delta_{x-y,a} \qquad \text{(A.17)}$$

the right hand side of Eq. (A.15) is rewritten as

$$\det\left[\cdots\left|2\cos(k_j a)\bar{A}_j(x_l-y)e^{k_j x_l}\right|\cdots\left|e^{\imath k_j y}\right.\right] \qquad \text{(A.18)}$$

$$+ \quad \det\left[\cdots\left|2\imath c\delta_{x_l-y,0}\sin(k_j a)e^{k_j x_l}\right|\cdots\left|e^{\imath k_j y}\right.\right]$$
$$+ \quad \det\left[\cdots\left|c\left(\delta_{x_l-y,-a}e^{\imath k_j a}-\delta_{x_l-y,a}e^{-\imath k_j a}\right)e^{k_j x_l}\right|\cdots\left|e^{\imath k_j y}\right.\right] \ .$$

Note that the last term in the expression above vanishes, since, due to the Kronecker-$\delta$, two columns coincide. Hence the kinetic contribution for the particles of the Fermi-sea yields

$$\sum_{l=1}^{N}\sum_{s=\pm1}\varphi(x_1,\ldots,x_l+sa,\ldots,x_N,y)=\sum_{l=1}^{N}\det\left[\cdots\left|2\cos(k_j a)\bar{A}_j(x_l-y)e^{k_j x_l}\right|\cdots\left|e^{\imath k_j y}\right.\right]$$
$$-2c\sum_{l=1}^{N}\delta_{x_l-y,0}\det\left[\cdots\left|\imath\sin(k_j a)e^{k_j x_l}\right|\cdots\left|e^{\imath k_j y}\right.\right] \ . \tag{A.19}$$

We turn to the kinetic term for the spin-up particle. By means of Eqs. (A.16) and (A.17) we obtain

$$\sum_{s=\pm1}\varphi(x_1,\ldots,,x_N,y+sa)=\det\left[(\bar{A}_j(x_l-y)-c\,(\delta_{x_l,y}+\delta_{x_l,y+a}))e^{\imath k_j x_l}\right|\left.e^{\imath k_j(y+a)y}\right]$$
$$+\det\left[(\bar{A}_j(x_l-y)+c\,(\delta_{x_l,y}+\delta_{x_l,y-a}))e^{\imath k_j x_l}\right|\left.e^{\imath k_j(y-a)y}\right] \ . \tag{A.20}$$

Again the terms with $\delta_{x_l,y\pm a}$ do not contribute, since two columns coincide. For the same reason only terms linear in $\delta_{x_l,y}$ will survive when the columns are expanded. Hence the right hand side of Eq. (A.20) acquires the form

$$\det\left[\bar{A}_{jl}(x_l-y)e^{\imath k_j x_l}\right|\left.2\cos(k_j a)e^{\imath k_j y}\right]+\sum_{l=1}^{N}2\imath c\delta_{x_l,y}\det\left[..\left|e^{\imath k_j x_l}\right|..\left|\sin(k_j a)e^{\imath k_j y}\right.\right] . \tag{A.21}$$

Combining the expressions (A.19) and (A.21), the action of the kinetic part of the Hamiltonian (2.62) yields

$$-t\left(\sum_{j=1}^{N}\sum_{s=\pm1}\varphi(x_1,\ldots,x_j+as,\ldots,x_N,y)+\varphi(x_1,\ldots,x_N,y\pm as)\right)=-t \tag{A.22}$$
$$\sum_{l=1}^{N}\det\left[..\left|2\cos(k_j a)\bar{A}_j(x_l-y)e^{\imath k_j x_l}\right|..\left|e^{\imath k_j y}\right.\right]-t\det\left[\bar{A}_{jl}(x_l-y)e^{\imath k_j x_l}\right|\left.2\cos(k_j a)e^{\imath k_j y}\right]$$
$$-4ct\sum_{l=1}^{N}\delta_{x_l,y}\det\left[\cdots\left|\imath\sin(k_j a)e^{\imath k_j x_l}\right|\cdots\left|e^{\imath k_j y}\right.\right] \ .$$

On the other hand, the action of the potential onto the wave function yields

$$U\sum_{l=1}^{N}\delta_{x_l,y}\varphi(x_1,\ldots,x_N,y)=\frac{U}{a}\sum_{l=1}^{N}\delta_{x_l,y}\det\left[\cdots\left|\imath\sin(k_j a)e^{\imath k_j x_l}\right|\cdots\left|e^{\imath k_j y}\right.\right] \ . \tag{A.23}$$

Choosing $c = U/(4at)$, we see that this contribution is canceled by the term in the third line of Eq. (A.22). The terms in the second line of Eq. (A.22) yield the energy. To reveal this we use

$$\sum_{l=1}^{N} \det\left[Q_{j1}\Big|..\Big|B_j Q_{jl}\Big|..\Big|Q_{jN}\right]_{j=1,\ldots,N} = \sum_{l=1}^{N} \sum_{\omega\in S(N)} \mathrm{sgn}(\omega) \prod_{j\neq l}^{N} Q_{\omega(j)j} Q_{\omega(l)l} B_{\omega(l)} \quad \text{(A.24)}$$

$$= \sum_{\omega\in S(N)} \mathrm{sgn}(\omega) \prod_{j=1}^{N} Q_{\omega(j)j} \left(\sum_{l=1}^{N} B_{\omega(l)}\right) = \left(\sum_{l=1}^{N} B_l\right) \det\left[Q_{jl}\right]_{j,l=1,\ldots,N},$$

where $Q_{jl}$ $j,l = 1,\ldots,N$ denotes an arbitrary $N\times N$ matrix and $B_j$ is a $N$-component vector. Employing the relation (A.24) shows that the second line of (A.22) yields the wave function times the factor $-2t\sum_{l=1}^{N+1}\cos(k_l a)$. This proofs Eq. (2.62) to be a solution to the difference equation (2.62) to the eigenvalue (2.69).

## A.3 Evaluation of the matrix entries $K_{jl}$

We evaluate the quantities $K_{jl}$ as defined in Eq. (3.41). Using the functional equation for the sign-function

$$\mathrm{sgn}(x-y)\mathrm{sgn}(x-y') = 1 + \mathrm{sgn}(y-y')(\mathrm{sgn}(x-y) - \mathrm{sgn}(x-y'))\ , \quad \text{(A.25)}$$

the integral in Eq. (3.41) is rewritten as

$$\left((k_j-\Lambda)(k_l-\Lambda)+c^2\right) \int_0^L dx e^{\imath(k_j-k_l)x} \quad \text{(A.26)}$$

$$+\quad c\left(\imath(k_j-\Lambda) - c\mathrm{sgn}(y-y')\right) \int_0^L dx\ \mathrm{sgn}(x-y') e^{\imath(k_j-k_l)x} \quad \text{(A.27)}$$

$$-\quad c\left(\imath(k_l-\Lambda) - c\mathrm{sgn}(y-y')\right) \int_0^L dx\ \mathrm{sgn}(x-y) e^{\imath(k_j-k_l)x}\ . \quad \text{(A.28)}$$

For $j = l$ the evaluation of this expression leads to

$$K_{jj} \quad = \quad \left((k_j-\Lambda)^2 + c^2\right) L + 2c\Big(\imath(k_j-\Lambda) - c\mathrm{sgn}(y-y')\Big)(y-y')\ . \quad \text{(A.29)}$$

For $j \neq l$ we use

$$\int_0^L dx\ \mathrm{sgn}(x-y) e^{\imath(k_j-k_l)x} = \frac{1}{\imath(k_j-k_l)}\left(1 + e^{\imath(k_j-k_l)L} - 2e^{\imath(k_j-k_l)y}\right) \quad \text{(A.30)}$$

and analogously for the integral with $y'$. Combining the two expression yields for $j \neq l$

$$\begin{aligned} K_{jl} &= \text{B.T.} - \frac{2c}{\imath(k_j-k_l)}\Big(\imath k_j e^{\imath(k_j-k_l)y'} - \imath k_l e^{\imath(k_j-k_l)y} \\ &\quad +\imath\Lambda\left(e^{\imath(k_j-k_l)y} - e^{\imath(k_j-k_l)y'}\right) + c\,\mathrm{sgn}(y-y')\left(e^{\imath(k_j-k_l)y} - e^{\imath(k_j-k_l)y'}\right)\Big) \ , \end{aligned} \tag{A.31}$$

where we have collected the boundary terms arising from the integration in

$$\text{B.T.} = \left((k_j-\Lambda)(k_l-\Lambda)+c^2\right)\frac{e^{\imath(k_j-k_l)L}-1}{\imath(k_j-k_l)} + ce^{\imath(k_j-k_l)L} + 1 \ . \tag{A.32}$$

We use the notation $y^{\pm} = (y \pm y')$. By expressing the exponentials in Eq. (A.31) in terms of sine and cosine we obtain

$$\begin{aligned} K_{jl} &= \text{B.T.} - 2ce^{\imath(k_j-k_l)y^+}\left(\cos\left((k_j-k_l)y^-/2\right)\right. \\ &+ \left.\frac{k_j+k_l-2\Lambda}{\imath(k_j-k_l)}\sin\left((k_j-k_l)y^-/2\right) + 2c\frac{\sin\left((k_j-k_l)|y^-|/2\right)}{k_j-k_l}\right) \ . \end{aligned} \tag{A.33}$$

We discuss the boundary term B.T. From the Bethe-Ansatz equations (3.4) we deduce

$$e^{\imath(k_j-k_l)L} = \frac{[\imath(k_j-\Lambda)-c]}{[\imath(k_j-\Lambda)+c]}\frac{[\imath(k_j-\Lambda)+c]}{[\imath(k_j-\Lambda)-c]} \ . \tag{A.34}$$

Hence it follows

$$e^{\imath(k_j-k_l)L} - 1 = \frac{2c\imath(k_j-k_l)}{[\imath(k_j-\Lambda)+c]\,[\imath(k_l-\Lambda)-c]} \ , \tag{A.35}$$

$$e^{\imath(k_j-k_l)L} + 1 = -\frac{2\left[(k_j-\Lambda)(k_l-\Lambda)+c^2\right]}{[\imath(k_j-\Lambda)+c]\,[\imath(k_l-\Lambda)-c]} \ . \tag{A.36}$$

Using Eqs. (A.35) and (A.36) reveals that B.T. = 0. Then combining the expressions Eq. (A.29) and (A.33) leads to the form of $K_{jl}$ in Eq. (3.42).

## A.4 Representation as Töplitz determinant

For finite interaction strength the determinant in Eq. (3.45) representing the Green's function is not a Töplitz determinant due to the non constant diagonal entries. However, in the limit $c \to \infty$ the Green's function has a representation as Töplitz determinant. To reveal it we write the quantities $g_{jl}$ in Eq. (3.44) as

$$\lim_{c\to\infty} g_{jl} = Lc^2 e^{-\imath(n_j+n_l+1+2\arctan(\lambda)/\pi)t}\left((\lambda^2+1)\delta_{jl} - 2(1+\imath\lambda)\frac{\sin\left(n_{jl}t\right)}{\pi n_{jl}}\right) \ . \tag{A.37}$$

Here we use the notations

$$t = \frac{\pi}{L}(y-y') \geq 0 \quad , \quad n_{jl} = j-l \tag{A.38}$$

and furthermore assume that the quantum numbers $n_j$ are given by the set in Eq. (3.9). Now the diagonal entries are constant, i.e. independent of $k_j$. For the linear combination of the quantities $g_{nm}$ as it appears in Eq. (3.46) this yields

$$\lim_{c\to\infty}\Big(g_{jl}-g_{jl+1}-g_{j+1l}+g_{j+1l+1}\Big)=Lc^2e^{-\imath(n_j+n_l+1+2\arctan(\lambda)/\pi)t}\gamma_{jl}(t,\lambda)\ , \quad \text{(A.39)}$$

where we have defined

$$\begin{aligned}\gamma_{jl}(t,\lambda) &= (\lambda^2+1)\Big(2\cos(t)\delta_{jl}-\delta_{jl+1}-\delta_{jl-1}\Big)-\frac{4}{\pi}(\imath\lambda+1)\frac{\sin(n_{jl}|t|)\cos(t)}{n_{jl}}\\ &+ \frac{2}{\pi}(\imath\lambda+1)\left(\frac{\sin((n_{jl}-1)|t|)}{n_{jl}-1}+\frac{\sin((n_{jl}+1)|t|)}{n_{jl}+1}\right)\ . \quad \text{(A.40)}\end{aligned}$$

The full Green's function as defined in Eq. (3.46) then reads

$$\lim_{c\to\infty}G(t)\propto e^{2\imath\arctan(\lambda)t/\pi}\det\left[\gamma_{jl}(t,\lambda)\right]_{j,l=1,\ldots,N}\ . \quad \text{(A.41)}$$

Employing the normalization condition $G(0)=1/L$ this evaluates to

$$\lim_{c\to\infty}G(t)=\frac{e^{2\imath\arctan(\lambda)t/\pi}}{L(N+1)(\lambda^2+1)^N}\det\left[\gamma_{jl}(t,\lambda)\right]_{j,l=1,\ldots,N}\ . \quad \text{(A.42)}$$

Correspondingly the interaction part $G_{\mathrm{I}}(t)$ reads

$$\lim_{c\to\infty}G_{\mathrm{I}}(t)=\frac{e^{-2\imath N\arctan(\lambda)t/\pi}}{(N+1)(\lambda^2+1)^N}\det\left[\gamma_{jl}(t,\lambda)\right]_{j,l=1,\ldots,N}\ . \quad \text{(A.43)}$$

For $\lambda=0$, Eq. (A.42) is equivalent to the representation of the Green's function for hardcore Bosons as Töplitz-determinant [56, 53]. If on the other hand $\lambda\to\infty$, the determinant in Eq. (A.42) becomes tridiagonal

$$\lim_{\lambda\to\infty}\lim_{c\to\infty}G(t)=\frac{e^{\imath t}}{L(N+1)}\underbrace{\det\left[(2\cos(t)\delta_{j,l}-\delta_{j,l+1}-\delta_{j,l-1})\right]_{j,l=1,\ldots,N}}_{\mathcal{L}_N(t)}\ . \quad \text{(A.44)}$$

Expanding it yields the following recursion relation

$$\mathcal{L}_N(t)=2\cos(t)\mathcal{L}_{N-1}(t)-\mathcal{L}_{N-2}(t)\ . \quad \text{(A.45)}$$

With the aid of the relation $2\cos(x)\sin(x)=\sin(x-y)+\sin(x+y)$, we see that its solution is given by

$$\mathcal{L}_N(t)=\frac{\sin((N+1)t)}{\sin(t)}\ . \quad \text{(A.46)}$$

Consequently Eq. (A.44) acquires the form

$$\lim_{\lambda\to\infty}\lim_{c\to\infty}G(t)=\frac{e^{\imath t}}{L(N+1)}\frac{\sin((N+1)t)}{\sin(t)}\ . \quad \text{(A.47)}$$

Analogously one obtains for the interaction part

$$\lim_{\lambda\to\infty}\lim_{c\to\infty}G_{\mathrm{I}}(t)=\frac{e^{-\imath Nt}}{N+1}\frac{\sin((N+1)t)}{\sin(t)}\ . \quad \text{(A.48)}$$

Equations (A.47) and (A.48) correspond to the free Fermion result.

## A.5 Two particle density-density correlation function

According to Theorem 1 the eigenfunctions acquire for $M = 2$ the form

$$\Psi(\mathbf{x}, \mathbf{k}, y_1, y_2, \Lambda_1, \Lambda_2) \propto \sum_{R \in S(2)} \mathrm{sgn}(R) \left[\imath(\Lambda_{R1} - \Lambda_{R2}) + 2c\mathrm{sgn}(y_2 - y_1)\right] \Phi(\mathbf{x}, \mathbf{k}, y_1, y_2, \Lambda_{R1}, \Lambda_{R2}) \ . \tag{A.49}$$

We recall that

$$\Phi(\mathbf{x}, \mathbf{k}, y_1, y_2, \Lambda_{R1}, \Lambda_{R2}) = \det \Big[ A_j(\Lambda_{R1}, x_l - y_1) A_j(\Lambda_{R2}, x_l - y_2) e^{\imath k_j x_l} \Big| \Big| A_j(\Lambda_{R2}, y_1 - y_2) e^{\imath k_j y_1} \Big| A_j(\Lambda_{R1}, y_2 - y_1) e^{\imath k_j y_2} \Big]_{\substack{j=1,\ldots,N+2 \\ l=1,\ldots,N}} \tag{A.50}$$

and

$$A_j(\Lambda, x) = \imath(k_j - \Lambda) + c\mathrm{sgn}(x) \ . \tag{A.51}$$

To unburden the notation we use in Secs. A.5 and A.6 the following convention: Whenever the second argument of $A_j(\Lambda, x)$ is dropped, the argument of the sign-function in Eq. (A.51) is positive i.e.

$$A_j(\Lambda) \equiv A_j(\Lambda, 1) = \imath(k_j - \Lambda) + c \ . \tag{A.52}$$

We consider Eq. (4.19) for $n = 0$. To evaluate it we take the expression (A.50) for $\Phi(\mathbf{x}, y_1, y_2, \Lambda_{R1}, \Lambda_{R2})$ as our starting point. Using translational invariance, we shift the integration variables in Eq. (4.19) by $x_l \to x_l + y_1$ and expand the determinant in Eq. (A.50) with respect to the last two columns. This yields

$$\Phi(\mathbf{x} - y_1, y_1, y_2, \Lambda_{R1}, \Lambda_{R2}) = \prod_{j=1}^{N+2} e^{\imath k_j y_1} \sum_{n \neq m}^{N+2} (-1)^{n+m} A_n(\Lambda_{R1}) A_m^*(\Lambda_{R1}) e^{\imath k_n y^-} \times \det \Big[ A_j(\Lambda_{R1}, x_l) A_j(\Lambda_{R2}, x_l - (y_2 - y_1)) e^{\imath k_j x_l} \Big]_{\substack{j=1,\ldots,N+2 \neq n,m \\ l=1,\ldots,N}} \tag{A.53}$$

with $y^- = y_2 - y_1$. Then by employing properties of the determinant, the integral in Eq. (4.19) can be cast into the form

$$\int_0^L dx_1 \cdots \int_0^L dx_N \Phi(\mathbf{x}, y_1, y_2, \Lambda_{R1}, \Lambda_{R2}) \Phi^*(\mathbf{x}, y_1, y_2, \Lambda_{R'1}, \Lambda_{R'2}) = N! \sum_{n \neq m}^{N+2} \sum_{s \neq t}^{N+2} (-1)^{n+m} (-1)^{s+t} \prod_{j=1}^{N+2} A_j(\Lambda_{R1}) A_j^*(\Lambda_{R'1}) \frac{A_m^*(\Lambda_{R2}) A_t(\Lambda_{R'2})}{A_m(\Lambda_{R1}) A_t^*(\Lambda_{R'1})} e^{\imath(k_n - k_s)y^-} \det \left[ Q_{jl} \right]_{\substack{j,l=1,\ldots,N+2 \\ j \neq n,m \ \ l \neq s,t}} , \tag{A.54}$$

where the quantities $Q_{jl}$ in Eq. (A.54) are given by

$$Q_{jl} = \int_0^L dx A_j(\Lambda_{R2}, x - y^-) A_l^*(\Lambda_{R'2}, x - y^-) e^{\imath(k_j - k_l)x} \ . \tag{A.55}$$

The integral is elementary. Evaluating it we obtain

$$\begin{aligned} Q_{jl} = & \left[ L A_j(\Lambda_{R2}) A_l^*(\Lambda_{R'2}) + 2c\imath \left(\Lambda_{R2} - \Lambda_{R'2}\right) y^- \right] \delta_{jl} \\ & -2c \left(1 - \frac{\Lambda_{R2} - \Lambda_{R'2}}{k_j - k_l}\right) e^{\imath(k_j - k_l)y^-} (1 - \delta_{jl}) + \text{B.T.} \ , \end{aligned} \tag{A.56}$$

where B.T. stands for the expression

$$\begin{aligned} \text{B.T.} \ = \ & \frac{1}{\imath(k_j - k_l)} \Big[ \left( (k_j - \Lambda_{R2}) (k_l - \Lambda_{R'2}) + c^2 \right) \left( e^{\imath(k_j - k_l)L} - 1 \right) \\ + \ & \imath c \left( k_j - k_l - \Lambda_{R2} + \Lambda_{R'2} \right) \left( e^{\imath(k_j - k_l)L} + 1 \right) \Big] (1 - \delta_{jl}) \ , \end{aligned} \tag{A.57}$$

which comprises the terms that arise from the boundaries when the off-diagonal terms with $j \neq l$ in Eq. (A.55) are integrated. However, using the Bethe-Ansatz equations it is shown that these terms vanish identically as a consequence of translational invariance. The expression (A.56) reveals that the diagonal terms where $j = l$ scale like $L$, while the off-diagonal terms scale like $c$. Therefore in the thermodynamic limit and when $c$ varies on the scale of $k_\text{F}$, the off-diagonal terms are negligible and the entries can be approximated by the diagonal terms only. In leading order of $L$ we find

$$\det \left[ Q_{jl} \right]_{\substack{j=1,\ldots,N+2 \neq n,m \\ l=1,\ldots,N+2 \neq s,t}} = \left( \prod_{j \neq n,m}^{N+2} L A_j(\Lambda_{R2}) A_j^*(\Lambda_{R'2}) \right) \det \begin{bmatrix} \delta_{ns} & \delta_{nt} \\ \delta_{ms} & \delta_{mt} \end{bmatrix} . \tag{A.58}$$

Combining the expression (A.58) with Eq. (A.54), the two particle density-density correlation function acquires the form

$$\begin{aligned} R_0(y_1, y_2) \propto & \sum_{R, R' \in S(2)} \operatorname{sgn}(R + R') \left[ \imath(\Lambda_{R1} - \Lambda_{R2}) + 2c \right] \left[ -\imath(\Lambda_{R'1} - \Lambda_{R'2}) + 2c \right] \\ & \sum_{n \neq m}^{N+2} \sum_{s \neq t}^{N+2} \frac{A_m^*(\Lambda_{R2}) A_t(\Lambda_{R'2}) e^{\imath(k_n - k_s)(y_1 - y_2)}}{A_m(\Lambda_{R1}) A_t^*(\Lambda_{R'1}) A_n(\Lambda_{R2}) A_m(\Lambda_{R2}) A_n^*(\Lambda_{R'2}) A_m^*(\Lambda_{R'2})} \det \begin{bmatrix} \delta_{ns} & \delta_{nt} \\ \delta_{ms} & \delta_{mt} \end{bmatrix} . \end{aligned} \tag{A.59}$$

Since the terms in the second line of Eq. (A.59) factorize in all summation indices, each term can be written into the corresponding row or column of the determinant. Furthermore, due to the Kronecker-$\delta$'s two of the four summations drop out. Thus the second line of the equation above can be expressed by a determinant whose entries are one-fold sums. Now it is straight forward to take the thermodynamic limit. Assuming the Fermi-sea to be in the ground state i.e. at zero temperature, the quasi-momenta

distribute themselves with the density $\varrho(k)$ given in Eq. (2.49). In the usual manner we replace the sums in Eq. (A.59) by integrals over the quasi-momenta. Then doing the summations over $R$ and $R'$, the leading order term of Eq. (A.59) can after some algebra be written as

$$R_0(y_1,y_2) \propto \left\{ \det[I^{(0)}(\hat{\Lambda}_1,\hat{\Lambda}_2)] - \mathrm{Re}\left( e^{2\imath \arctan\left(\frac{\Lambda_1-\Lambda_2}{2c}\right)} \det[J^{(0)}(\hat{\Lambda}_1,\hat{\Lambda}_2)] \right) \right\} \ . \quad \text{(A.60)}$$

From the particular form of Eq. (A.59) it follows that $I^{(0)}(\hat{\Lambda}_1,\hat{\Lambda}_2)$ and $J^{(0)}(\hat{\Lambda}_1,\hat{\Lambda}_2)$ are matrices with dimension $2\times 2$. Furthermore, the entries can be identified from Eq. (A.59) to be give by Eq. (4.24).

It remains to evaluate the normalization constant. The normalization condition reads

$$4 = \int_0^L dy_1 \int_0^L dy_2 \ R_0(y_1,y_2) \ . \quad \text{(A.61)}$$

To evaluate it, we use the form (A.59) of $R_0(y_1,y_2)$. The integration over the exponential there yields in leading order of $L$

$$\int_0^L dy_1 \int_0^L dy_2 e^{\imath(k_n-k_s)(y_2-y_1)} = L^2\delta_{ns} \ . \quad \text{(A.62)}$$

Thus only the diagonal terms of the entries (4.24) contribute to the normalization. This immediately leads to Eq. (4.22).

## A.6 Three particle density-density correlation function

The derivation of the three particle density-density correlation function follows along the same lines as in Sec. A.5. We consider the expression (4.19) for $n=1$. Expanding the determinant in (A.50) with respect to the last three columns it acquires the form

$$\Phi(\mathbf{x},y_1,y_2,\Lambda_{R1},\Lambda_{R2}) = \quad \text{(A.63)}$$
$$\sum_{\alpha\neq\beta\neq\gamma}^{N+2} (-1)^{\alpha+\beta+\gamma+1} e^{\imath k_\alpha y_1 + \imath k_\beta y_2 + \imath k_\gamma x} A_\gamma(\Lambda_{R1},x-y_1) A_\gamma(\Lambda_{R2},x-y_2) A_\beta(\Lambda_{R1}) A_\alpha^*(\Lambda_{R2})$$
$$\det\left[ A_j(\Lambda_{R1},x_l-y_1) A_j(\Lambda_{R2},x_l-y_2) e^{\imath k_j x_l} \right]_{\substack{j=1,\ldots,N+2\neq\alpha,\beta,\gamma \\ l=1,\ldots,N-1}} \ ,$$

where we use the notation $x_N \equiv x$ and we assume $y_1 \leq y_2$. For the integral in the last line of Eq. (4.19) this yields

$$\int_0^L dx_2 \ldots \int_0^L dx_N \Phi(\mathbf{x},y_1,y_2,\Lambda_{R1},\Lambda_{R2}) \Phi^*(\mathbf{x},y_1,y_2,\Lambda_{R'1},\Lambda_{R'2}) \quad \text{(A.64)}$$

$$= \sum_{\substack{\alpha\neq\beta\neq\gamma \\ \mu\neq\nu\neq\sigma}}^{N+1} (-1)^{\alpha+\beta+\gamma+\mu+\nu+\sigma} e^{\imath(k_\alpha-k_\mu)y_1+\imath(k_\beta-k_\nu)y_2+\imath(k_\gamma-k_\sigma)x} \det\left[Q_{jl}\right]_{\substack{j\neq\alpha,\beta,\gamma \\ l\neq\mu,\nu,\sigma}}$$

$$A_\alpha(\Lambda_{R2})A^*_\beta(\Lambda_{R1})A^*_\mu(\Lambda_{R'2})A_\nu(\Lambda_{R'1}) \prod_{\substack{j=1 \\ \neq\alpha,\beta,\gamma}}^{N+2} A_j(\Lambda_{R2})e^{\imath k_j y_2} \prod_{\substack{j=1 \\ \neq\mu,\nu,\gamma}}^{N+2} A^*_j(\Lambda_{R2})e^{-\imath k_j y_2}$$

$$A_\gamma(\Lambda_{R1}, x-y_1)A_\gamma(\Lambda_{R2}, x-y_2)A^*_\sigma(\Lambda_{R'1}, x-y_1)A^*_\sigma(\Lambda_{R'2}, x-y_2)\ .$$

The entries $Q_{jl}$ of the determinant are given by Eq. (A.55). Following the discussion there, we have in leading order of $L$

$$\det\left[Q_{jl}\right]_{\substack{j=1,\ldots,N+2\neq\alpha,\beta,\gamma \\ l=1,\ldots,N+2\neq\mu,\nu,\sigma}} = \prod_{\substack{j=1 \\ \neq\alpha,\beta,\gamma}}^{N+2} L^2|A_j(\Lambda_{R1})|^2 \det\begin{bmatrix} \delta_{\alpha\mu} & \delta_{\alpha\nu} & \delta_{\alpha\sigma} \\ \delta_{\beta\mu} & \delta_{\beta\nu} & \delta_{\beta\sigma} \\ \delta_{\gamma\mu} & \delta_{\gamma\nu} & \delta_{\gamma\sigma} \end{bmatrix}. \quad \text{(A.65)}$$

Substituting Eq. (A.65) into Eq. (A.64), the three particle density-density correlation function acquires the form

$$R_1(y_1, y_2, x) \propto \sum_{R,R'\in S(2)} \mathrm{sgn}(R+R')\left[\imath(\Lambda_{R1}-\Lambda_{R2})+2c\right]\left[-\imath(\Lambda_{R'1}-\Lambda_{R'2})+2c\right] \quad \text{(A.66)}$$

$$\sum_{\substack{\alpha\neq\beta\neq\gamma \\ \mu\neq\nu\neq\sigma}}^{N+1} \frac{e^{\imath(k_\alpha-k_\mu)y_1+\imath(k_\beta-k_\nu)y_2+\imath(k_\gamma-k_\sigma)x}}{\prod_{j=1}^2 |A_\alpha(\Lambda_{Rj})|^2|A_\beta(\Lambda_{Rj})|^2|A_\gamma(\Lambda_{Rj})|^2} \det\begin{bmatrix} \delta_{\alpha\mu} & \delta_{\alpha\nu} & \delta_{\alpha\sigma} \\ \delta_{\beta\mu} & \delta_{\beta\nu} & \delta_{\beta\sigma} \\ \delta_{\gamma\mu} & \delta_{\gamma\nu} & \delta_{\gamma\sigma} \end{bmatrix} \quad \text{(A.67)}$$

$$A^*_\alpha(\Lambda_{R2})A_\beta(\Lambda_{R1})A_\gamma(\Lambda_{R1}, x-y_1)A_\gamma(\Lambda_{R2}, x-y_2) \quad \text{(A.68)}$$

$$A_\mu(\Lambda_{R'2})A^*_\nu(\Lambda_{R'1})A^*_\sigma(\Lambda_{R'1}, x-y_1)A^*_\sigma(\Lambda_{R'2}, x-y_2)\ . \quad \text{(A.69)}$$

This is the analog expression to (A.59) for the three particle density-density correlation function. Due to the Kronecker-$\delta$'s, three of the six summations drop out. Similarly to the discussion after Eq. (A.59), we write the terms in lines (A.67)-(A.69) in the determinant and do the thermodynamic limit. The summation over $R$ and $R'$ can be carried out and in perfect analogy to the two particle case, the three particle density-density correlation function can be cast into the form

$$R_1(y_1, y_2, x) \propto \left\{ \det[I^{(1)}(\hat{\Lambda}_1, \hat{\Lambda}_2)] - \mathrm{Re}\left( e^{2\imath \arctan\left(\frac{\Lambda_1-\Lambda_2}{2c}\right)} \det[J^{(1)}(\hat{\Lambda}_1, \hat{\Lambda}_2)] \right) \right\}\ , \quad \text{(A.70)}$$

Due to Eq. (A.66), the matrices $I^{(1)}(\hat{\Lambda}_1, \hat{\Lambda}_2)$ and $J^{(1)}(\hat{\Lambda}_1, \hat{\Lambda}_2)$ have now dimension $3\times 3$. As can be deduced from (A.66) the corresponding entries are given by Eqs. (4.29) and (4.30).

We turn to the normalization. Now the normalization condition reads

$$4N = \int_0^L dx \int_0^L dy_1 \int_0^L dy_2\ R_1(y_1, y_2, x)\ . \quad \text{(A.71)}$$

To evaluate Eq. (A.71), we use the form (A.66) of $R_1(y_1, y_2, x)$. The integrals are carried out by employing translational invariance in order shift, first $x \to x + y_2$ and afterwards $y_1 \to y_1 + x + y_2$. Then all pre-exponential terms in Eq. (A.66) become coordinate independent. The integrals over the exponentials yield in leading order of $L$

$$\int_0^L dx \int_0^L dy_1 \int_0^L dy_2 e^{\imath(k_\alpha - k_\mu)y_1 + \imath(k_\beta - k_\nu)y_2 + \imath(k_\gamma - k_\sigma)x} = L^3 \delta_{\alpha\mu}\delta_{\beta\nu}\delta_{\gamma\sigma} \; . \tag{A.72}$$

Again only the diagonal terms (4.29) and (4.30) contribute to the normalization. This immediately leads to the normalization constant (4.22) and together with Eq. (A.70) to the form (4.21) of the three particle density-density correlation function.

# Appendix B

# Derivations for Part II

## B.1 Commutator relation of the statistical function

We proof that the statistical functions fulfill the relations

$$\left[H_{N+1,M}(\mathbf{z}') - H_{N,M}(\mathbf{z}), I^{\dagger}_{N\downarrow} I^{\dagger}_{M\uparrow}\right] a^{\dagger}_{k} = 0 \,, \tag{B.1}$$

$$\left[H_{N-1,M}(\mathbf{z}'') - H_{N,M}(\mathbf{z}), I_{N\downarrow} I_{M\uparrow}\right] a_{k} = 0 \,, \tag{B.2}$$

where the Hamiltonian is defined by Eq. (5.16). The statistical function $I^{(\dagger)}_{N\downarrow} I^{(\dagger)}_{M\uparrow}$ and the symmetric annihilation (creation) function $a^{(\dagger)}_{k}$ are determined via Eqs. (5.24)-(5.27) and by Eqs. (5.39)-(5.40). To unburden the notation we drop the coordinate dependence of the creation and annihilation functions as well as of the statistical functions. We consider the relation (B.1).

The potential terms in the Hamiltonians $H_{N+1,M}(\mathbf{z}')$, $H_{N,M}(\mathbf{z})$ commute with the statistical function. Therefore it is sufficient to consider the commutator of the kinetic terms. It is then easily shown that the condition (B.1) yields

$$\begin{aligned} 0 = & \left( \frac{I^{\dagger}_{M\uparrow}}{2m_1} \left( \sum_{n=1}^{N+1} \frac{\partial^2}{\partial {x'_n}^2} - \sum_{l=1}^{N} \frac{\partial^2}{\partial x_l^2} \right) I^{\dagger}_{N\downarrow} a^{\dagger}_{k} \right. \\ & \left. + \frac{I^{\dagger}_{N\downarrow}}{2m_2} \left( \sum_{m=1}^{N+1} \frac{\partial^2}{\partial {y'_m}^2} - \sum_{l=1}^{M} \frac{\partial^2}{\partial y_l^2} \right) I^{\dagger}_{M\uparrow} a^{\dagger}_{k} \right) \,. \end{aligned} \tag{B.3}$$

We require the equation above to hold for arbitrary $m_1$ and $m_2$. This can only be true if the two terms on the right hand side vanish independently of each other. Doing the second derivatives we obtain the two conditions

$$\sum_{n=1}^{N+1} \left( a^{\dagger}_{k} \frac{\partial^2 I^{\dagger}_{N\downarrow}}{\partial {x'_n}^2} + 2 \frac{\partial I^{\dagger}_{N\downarrow}}{\partial x'_n} \frac{\partial a^{\dagger}_{k}}{\partial x'_n} \right) = \sum_{n=1}^{N} \left( a^{\dagger}_{k} \frac{\partial^2 I^{\dagger}_{N\downarrow}}{\partial x_n^2} + 2 \frac{\partial I^{\dagger}_{N\downarrow}}{\partial x_n} \frac{\partial a^{\dagger}_{k}}{\partial x_n} \right) \,, \tag{B.4}$$

$$\sum_{m=1}^{M} \left( a^{\dagger}_{k} \frac{\partial^2 I^{\dagger}_{M\uparrow}}{\partial {y'_m}^2} + 2 \frac{\partial I^{\dagger}_{M\uparrow}}{\partial y'_m} \frac{\partial a^{\dagger}_{k}}{\partial y'_m} \right) = \sum_{m=1}^{M} \left( a^{\dagger}_{k} \frac{\partial^2 I^{\dagger}_{M\uparrow}}{\partial y_m^2} + 2 \frac{\partial I^{\dagger}_{M\uparrow}}{\partial y_m} \frac{\partial a^{\dagger}_{k}}{\partial y_m} \right) \,. \tag{B.5}$$

Using the explicit form (5.24) of $I^{\dagger}_{N\downarrow}$ it is straight forward to show that the summation over the second derivatives of $I^{\dagger}_{N\downarrow}$ with respect to the primed and the unprimed variables coincide i.e.

$$\sum_{n=1}^{N+1} \frac{\partial^2 I^{\dagger}_{N\downarrow}}{\partial {x'_n}^2} = \sum_{l=1}^{N} \frac{\partial^2 I^{\dagger}_{N\downarrow}}{\partial x_l^2} \tag{B.6}$$

and analogously for $I^{\dagger}_{M\uparrow}$. The conditions (B.4) and (B.5) then become

$$\sum_{n=1}^{N+1} \frac{\partial I^{\dagger}_{N\downarrow}}{\partial x'_n} \frac{\partial a^{\dagger}_k}{\partial x'_n} = \sum_{n=1}^{N} \frac{\partial I^{\dagger}_{N\downarrow}}{\partial x_n} \frac{\partial a^{\dagger}_k}{\partial x_n} \quad , \quad \sum_{m=1}^{M} \frac{\partial I^{\dagger}_{M\uparrow}}{\partial y'_m} \frac{\partial a^{\dagger}_k}{\partial y'_m} = \sum_{m=1}^{M} \frac{\partial I^{\dagger}_{M\uparrow}}{\partial y_m} \frac{\partial a^{\dagger}_k}{\partial y_m} \; . \tag{B.7}$$

We consider the first condition of the equation above. Taking the derivative of $I^{\dagger}_{N\downarrow}$ with respect to the primed and the unprimed variables yields

$$\begin{aligned} 0 = \sum_{n=1}^{N+1} \sum_{j=1}^{N} \det \Big[ \operatorname{sgn}(x'_s - x_t) \Big| 1 \Big]_{\substack{s=1,\dots,N+1 \neq n \\ t=1,\dots,N \neq m}} \delta(x'_n - x_j) \left[ \left( \frac{\partial a^{\dagger}_k}{\partial x'_n} \right) + \left( \frac{\partial a^{\dagger}_k}{\partial x_j} \right) \right] \\ \iff \left( \frac{\partial a^{\dagger}_k}{\partial x'_n} \right) \Bigg|_{x'_n = x_j} = - \left( \frac{\partial a^{\dagger}_k}{\partial x_j} \right) \Bigg|_{x'_n = x_j} \; . \end{aligned} \tag{B.8}$$

The first derivatives in the equation above can be calculated using the definition of $a^{\dagger}_k$, see Eq. (5.39). This yields

$$\begin{aligned} \frac{\partial a^{\dagger}_k}{\partial x'_n} = a^{\dagger}_k \Bigg( & 2\imath m_1 k - \lambda_1 \sum_{i \neq n}^{N+1} f(x'_n - x'_i) + \lambda_1 \sum_{s=1}^{N} f(x'_n - x_s) \\ & - \lambda_3 \sum_{l=1}^{M} f(x'_n - y'_l) + \lambda_3 \sum_{l=1}^{M} f(x'_n - y_l) \Bigg) \; , \end{aligned} \tag{B.9}$$

$$\begin{aligned} \frac{\partial a^{\dagger}_k}{\partial x_j} = a^{\dagger}_k \Bigg( & -2\imath m_1 k + \lambda_1 \sum_{l=1}^{N+1} f(x_j - x'_l) - \lambda_1 \sum_{i \neq j}^{N} f(x_j - x_i) \\ & + \lambda_3 \sum_{l=1}^{M} f(x_j - y'_l) - \lambda_3 \sum_{l=1}^{M} f(x_j - y_l) \Bigg) \; . \end{aligned} \tag{B.10}$$

Evaluated at $x_j = x'_n$, this immediately leads to the first condition in Eq. (B.8) and thus shows that also condition (B.4) holds. Repeating the steps above reveals that likewise

$$\left( \frac{\partial a^{\dagger}_k}{\partial y_m} \right) \Bigg|_{y_m = y'_l} = - \left( \frac{\partial a^{\dagger}_k}{\partial y'_l} \right) \Bigg|_{y_m = y'_l} \tag{B.11}$$

is a sufficient condition for Eq. (B.5) to be true. The correctness of Eq. (B.11) in turn is proven by a direct calculation analog to the one give above. This completes proof of the commutator relation (B.1). The proof for the commutator relation (B.2) for the annihilation operator follows along the same lines.

## B.2 Proof of Theorem 2

We show that the creation and annihilation functions defined in Eqs. (5.39) and (5.40) are solutions to the differential Eqs. (5.33) and (5.34).

The first derivative of $a_k^\dagger$ with respect to the primed variables $x'_n$ and $y'_m$ yields

$$\begin{aligned}\frac{\partial a_k^\dagger}{\partial x'_n} &= a_k^\dagger \Bigg( 2\imath k m_1 - \lambda_1 \sum_{i\neq n}^{N+1} f(x'_n - x'_i) + \lambda_1 \sum_{j=1}^{N} f(x'_n - x_j) \\ &- \lambda_3 \sum_{j=1}^{M} (f(x'_n - y'_j) - f(x'_n - y_j)) \Bigg) \end{aligned} \tag{B.12}$$

$$\begin{aligned}\frac{\partial a_k^\dagger}{\partial y'_m} &= a_k^\dagger \Bigg( 2\imath k m_2 - \lambda_2 \sum_{j\neq m}^{M} f(y'_m - y'_j) + \lambda_2 \sum_{j=1}^{M} f(y'_m - y_j) \\ &- \lambda_3 \sum_{i=1}^{N+1} f(y'_m - x'_i) + \lambda_3 \sum_{i=1}^{N} f(y'_m - x_i) \Bigg). \end{aligned} \tag{B.13}$$

The derivatives with respect to the unprimed variables look similar. Doing the second derivatives is straight forward. However, the resulting expressions become more involved. Exemplarily we state the expression for the summation over the second derivatives with respect to the set $\mathbf{x}'$

$$\sum_{n=1}^{N+1} \frac{\partial^2}{\partial {x'}_n^2} a_k^\dagger = a_k^\dagger \left( -(N+1) 4 m_1^2 k^2 \right. \tag{B.14}$$

$$+ \sum_{n\neq i}^{N+1} \left( \lambda_1^2 f^2(x'_n - x'_i) - \lambda_1 f'(x'_n - x'_i) \right) + \lambda_1^2 \sum_{n\neq i\neq j}^{N+1} f(x'_n - x'_i) f(x'_n - x'_j) \tag{B.15}$$

$$+ \sum_{n=1}^{N+1} \sum_{j=1}^{M} \left( \lambda_3^2 f^2(x'_n - y'_j) - \lambda_3 f'(x'_n - y'_j) \right) + \lambda_3^2 \sum_{n=1}^{N+1} \sum_{j\neq l}^{M} f(x'_n - y'_j) f(x'_n - y'_l) \tag{B.16}$$

$$+ 2\lambda_1\lambda_3 \sum_{i\neq n}^{N+1} \sum_{l=1}^{M} f(x'_n - x'_i) f(x'_n - y'_l) - 4\imath k m_1 \lambda_3 \sum_{n=1}^{N+1} \sum_{j=1}^{M} f(x'_n - y'_j) \tag{B.17}$$

$$+ \lambda_1 \sum_{n=1}^{N+1} \sum_{j=1}^{N} f'(x'_n - x_j) + \lambda_1^2 \sum_{n=1}^{N+1} \sum_{j=1}^{N} f^2(x'_n - x_j) \tag{B.18}$$

$$+ 4\imath k m_1 \lambda_3 \sum_{n=1}^{N+1} \sum_{j=1}^{M} f(x'_n - y_j) + 4\imath k m_1 \lambda_1 \sum_{n=1}^{N+1} \sum_{j=1}^{N} f(x'_n - x_j) \tag{B.19}$$

$$+ 2\lambda_1\lambda_3 \left( \sum_{j=1}^{N} \sum_{l=1}^{M} \sum_{n=1}^{N+1} (f(x'_n - x_j) f(x'_n - y_l) - f(x'_n - x_j) f(x'_n - y'_l)) \right. \tag{B.20}$$

$$-\sum_{i\neq n}^{N+1}\sum_{l=1}^{M} f(x'_n - x'_i)f(x'_n - y_l)\Bigg) + \sum_{n=1}^{N+1}\sum_{j=1}^{M}(\lambda_3^2 f^2(x'_n - y_j) + \lambda_3 f'(x'_n - y_j)) \quad \text{(B.21)}$$

$$+\lambda_1^2\left(\sum_{n=1}^{N+1}\sum_{j\neq l}^{N} f(x'_n - x_j)f(x'_n - x_l) - 2\sum_{i\neq n}^{N+1}\sum_{j=1}^{N} f(x'_n - x_j)f(x'_n - x'_i)\right) \quad \text{(B.22)}$$

$$+\lambda_3^2\left(\sum_{n=1}^{N+1}\sum_{j\neq l}^{M} f(x'_n - y_j)f(x'_n - y_l) - 2\sum_{n=1}^{N+1}\sum_{j,l}^{M} f(x'_n - y_j)f(x'_n - y'_l)\right)\Bigg). \quad \text{(B.23)}$$

Expressions with a similar structure are obtained for the sums over the second derivatives with respect to the sets $\mathbf{x}$, $\mathbf{y}'$ and $\mathbf{y}$. We comment on how to combine these terms when acting with the second derivatives of $H_{N+1,M}(\mathbf{x}', \mathbf{y}')$ and $H_{N,M}(\mathbf{x}, \mathbf{y})$ on $a_k^\dagger$. The terms in lines (B.14)-(B.17) only depend on the sets $\mathbf{x}', \mathbf{y}'$ of primed variables. Together with the corresponding terms that result from the derivatives with respect to the set $\mathbf{y}$, they can be combined to an auxiliary Hamiltonian $\tilde{H}_{N+1,M}(\mathbf{x}', \mathbf{y}')$ which is introduced below (see Eq. (B.24)). Analogously the corresponding terms from the derivative with respect to the unprimed variables combine to $\tilde{H}_{N,M}(\mathbf{x}, \mathbf{y})$. Note that thereby the complex part i.e. the last term in line (B.17) drops out. The terms in lines (B.18) and (B.19) cancel with the corresponding terms of the second derivative with respect to the set $\mathbf{x}$. The same holds for the derivatives with respect to $\mathbf{y}'$ and $\mathbf{y}$. Finally the terms in lines (B.20)-(B.23) do not drop out. When combining these terms with the ones resulting from the remaining derivatives, the corresponding expressions will yield the condition on $f$ in form of a functional equation and relations between the parameters $m_1$, $m_2$, $\lambda_1$, $\lambda_2$ and $\lambda_3$ which are necessary in order for $a_k^\dagger$ to fulfill the differential equation (5.33).

We introduce the auxiliary Hamiltonian

$$\begin{aligned}
\tilde{H}_{N+1,M}(\mathbf{x}', \mathbf{y}') = &-\frac{1}{2m_1}\sum_{n=1}^{N+1}\frac{\partial^2}{\partial x'^2_n} - \frac{1}{2m_2}\sum_{m=1}^{M}\frac{\partial^2}{\partial y'^2_m} \quad \text{(B.24)}\\
+ &\frac{1}{2m_1}\sum_{n\neq j}^{N+1}\left(\lambda_1^2 f^2(x'_n - x'_j) - \lambda_1 f'(x'_n - x'_j)\right) +\\
+ &\frac{1}{2m_2}\sum_{j\neq l}^{M}\left(\lambda_2^2 f^2(y'_j - y'_l) - \lambda_2 f'(y'_j - y'_l)\right)\\
+ &\left(\frac{1}{2m_1} + \frac{1}{2m_2}\right)\sum_{n=1}^{N+1}\sum_{m=1}^{M}\left(\lambda_3^2 f^2(x'_n - y'_m) - \lambda_3 f'(x'_n - y'_m)\right)\\
+ &\frac{\lambda_1^2}{2m_1}\sum_{n\neq i\neq j}^{N+1} f(x'_n - x'_i)f(x'_n - x'_j) + \frac{\lambda_2^2}{2m_2}\sum_{m\neq j\neq l}^{M} f(y'_m - y'_j)f(y'_m - y'_l)\\
+ &\sum_{n=1}^{N+1}\sum_{j\neq l}^{M}\left(\frac{\lambda_3^2}{2m_1}f(x'_n - y'_j)f(x'_n - y'_l) + \frac{\lambda_2\lambda_3}{m_2}f(y'_j - y'_l)f(y'_j - x'_n)\right)
\end{aligned}$$

$$+ \sum_{m=1}^{M}\sum_{n\neq i}^{N+1}\left(\frac{\lambda_3^2}{2m_2}f(y'_m-x'_n)f(y'_m-x'_i)+\frac{\lambda_1\lambda_3}{m_1}f(x'_n-x'_i)f(x'_n-y'_m)\right) ,$$

which comprises all terms that depend only on one set (primed or unprimed) of variables. With help of Eq. (B.24) we have

$$\left[\tilde{H}_{N+1,M}(\mathbf{x}',\mathbf{y}')-\tilde{H}_{N,M}(\mathbf{x},\mathbf{y})\right]a_k^\dagger = 2m_1k^2a_k^\dagger \tag{B.25}$$

$$+\left[g(\mathbf{x}',\mathbf{x},\lambda_1)+g(\mathbf{y}',\mathbf{y},\lambda_2)\right. \tag{B.26}$$

$$-h(\mathbf{x}',\mathbf{x},\mathbf{y}',\mathbf{y},\lambda_1)+h(\mathbf{x},\mathbf{x}',\mathbf{y},\mathbf{y}',\lambda_1)-h(\mathbf{y}',\mathbf{y},\mathbf{x}',\mathbf{x},\lambda_2)+h(\mathbf{y},\mathbf{y}',\mathbf{x},\mathbf{x}',\lambda_2) \tag{B.27}$$

$$-\left(\frac{1}{2m_1}-\frac{1}{2m_2}\right)\left(\sum_{n=1}^{N+1}\sum_{j=1}^{M}(\lambda_3^2f^2(x'_n-y_j)+\lambda_3f'(x'_n-y_j))\right. \tag{B.28}$$

$$\left.\left.-\sum_{n=1}^{N}\sum_{j=1}^{M}(\lambda_3^2f^2(x_n-y'_j)+\lambda_3f'(x_n-y'_j))\right)\right]a_k^\dagger , \tag{B.29}$$

where we use the abbreviations

$$g(\mathbf{x}',\mathbf{x},\lambda_s)=\frac{\lambda_s^2}{2m_s}\left(\sum_{n=1}^{N+1}\sum_{j\neq l}^{N}[f(x'_n-x_j)f(x'_n-x_l)+2f(x_j-x_l)f(x_j-x'_n)]\right. \tag{B.30}$$

$$\left.-\sum_{n\neq i}^{N+1}\sum_{j=1}^{N}[2f(x'_n-x_j)f(x'_n-x'_i)+f(x_j-x'_n)f(x_j-x'_i)]\right)$$

$$h(\mathbf{x}',\mathbf{x},\mathbf{y}',\mathbf{y},\lambda_s)=\frac{\lambda_s\lambda_3}{m_s}\sum_{n=1}^{N+1}\sum_{j=1}^{N}\sum_{l=1}^{M}(f(x'_n-x_j)f(x'_n-y_l) , \tag{B.31}$$

$$-f(x'_n-x_j)f(x'_n-y'_l)-f(x'_n-x'_i)f(x'_n-y_l))$$

$$-\frac{\lambda_3^2}{2m_s}\left(\sum_{n=1}^{N+1}\sum_{j\neq l}^{M}f(x'_n-y_j)f(x'_n-y_l)-2\sum_{n=1}^{N+1}\sum_{j,l=1}^{M}f(x'_n-y_j)f(x'_n-y'_l)\right) .$$

For Eq. (B.25) to be of the form (5.33), the terms in the squared bracket of lines (B.26)-(B.29) have to become constant. A necessary condition for this to be the case, is that $f$ fulfills

$$f(x-y)f(x-z)+f(y-x)f(y-z)+f(z-x)f(z-y)=\text{const.} \tag{B.32}$$

This is the functional equation stated in Eq. (5.37) of Theorem 2. Its solution has been obtained in Ref. [67]. It reads

$$f(x)=z\lambda\coth(zx-\kappa). \tag{B.33}$$

Here $z$, $\lambda$ relate to the right hand side of Eq. (B.32) via const. $=\lambda^2z^2$ and $\kappa$ is an arbitrary complex constant. From Eq. (B.32) one concludes

$$f(x'_n-x_j)f(x'_n-x_l)+2f(x_j-x_l)f(x_j-x'_n)=\text{const.} \tag{B.34}$$

and consequently

$$g(\mathbf{x}',\mathbf{x},\lambda_1)=\frac{\lambda_1^2}{2m_1}N(N+1)\text{const.}\quad,\quad g(\mathbf{y}',\mathbf{y},\lambda_2)=0. \tag{B.35}$$

Next we consider the terms in line (B.27). By means of the functional equation we have

$$-h(\mathbf{x}',\mathbf{x},\mathbf{y}',\mathbf{y},\lambda_1)+h(\mathbf{x},\mathbf{x}',\mathbf{y},\mathbf{y}',\lambda_1)-h(\mathbf{y}',\mathbf{y},\mathbf{x}',\mathbf{x},\lambda_2)+h(\mathbf{y},\mathbf{y}',\mathbf{x},\mathbf{x}',\lambda_2) \tag{B.36}$$

$$=\text{const.}\left(-\frac{\lambda_1\lambda_3}{2m_1}MN(N-1)-\frac{\lambda_2\lambda_3}{2m_2}(N+1)M(M-1)+\frac{\lambda_2\lambda_3}{2m_2}M^2\right) \tag{B.37}$$

$$+\frac{\lambda_3}{2}\sum_{n\neq i}^{N+1}\sum_{l=1}^{M}\left(\frac{\lambda_1}{m_1}2f(x_n'-x_i')f(x_n'-y_j)+\frac{\lambda_3}{m_2}f(y_j-x_i')f(y_j-x_n')\right) \tag{B.38}$$

$$+\frac{\lambda_3}{2}\sum_{j\neq l}^{M}\sum_{n=1}^{N}\left(\frac{\lambda_2}{m_2}2f(y_j'-y_l')f(y_j'-x_n)+\frac{\lambda_3}{m_1}f(x_n-y_j')f(x_n-y_l')\right) \tag{B.39}$$

$$+\lambda_3\left(\frac{\lambda_3}{m_2}-\frac{\lambda_1}{m_1}\right)\left(\sum_{m=1}^{M}\sum_{i=1}^{N+1}\sum_{j=1}^{N}f(y_m'-x_i')f(y_m'-x_j)\right. \tag{B.40}$$

$$\left.-\sum_{m=1}^{M}\sum_{i\neq j}^{N+1}f(y_m-x_i')f(y_m-x_j')-\frac{1}{2}\sum_{m=1}^{M}\sum_{i\neq n}^{N}f(y_m'-x_i)f(y_m'-x_j)\right) \tag{B.41}$$

$$+\lambda_3\left(\frac{\lambda_3}{m_1}-\frac{\lambda_2}{m_2}\right)\left(\sum_{n=1}^{N+1}\sum_{j=1}^{M}\sum_{l=1}^{M}f(x_n'-y_j)f(x_n'-y_l')\right. \tag{B.42}$$

$$\left.-\sum_{n=1}^{N+1}\sum_{j\neq l}^{M}f(x_n-y_j')f(x_n-y_l')-\frac{1}{2}\sum_{n=1}^{N+1}\sum_{j\neq l}^{M}f(x_n'-y_j)f(x_n'-y_l)\right). \tag{B.43}$$

For this to become constant the functional equation (B.32) is not sufficient. In addition we require

$$\frac{\lambda_3}{m_2}=\frac{\lambda_1}{m_1}\qquad\text{and}\qquad\frac{\lambda_3}{m_1}=\frac{\lambda_2}{m_2}\ . \tag{B.44}$$

These are the first two conditions in Eq. (5.38). Provided that these relations hold, the terms in lines (B.38) and (B.39) become constant due to the functional equation and the terms in lines (B.40)-(B.43) vanish identically. Then the right hand side of Eq. (B.36) is cast into the form

$$\frac{\lambda_3^2}{m_2}NM\text{const.}+\frac{\lambda_3^2}{2m_1}M(M+1)\text{const.} \tag{B.45}$$

It remains to show that also the terms in lines (B.28)-(B.29) are constant. Two cases have to be distinguished. First, for $m_1=m_2$, it immediately follows that the terms in

lines (B.28)-(B.29) vanish. For $m_1 \neq m_2$ we take into account the solution Eq. (B.33) for $f$ to deduce

$$\lambda_3^2 f^2(x) + \lambda_3 f'(x) = \lambda_3 \left(\lambda_3 - \frac{1}{\lambda}\right) f^2(x) + \lambda\lambda_3 z^2 \tag{B.46}$$

This becomes constant for $\lambda = 1/\lambda_3$. Combining these two cases yields to the third condition in Eq. (5.38). With help of the relation const. $= \lambda^2 z^2$, the terms in lines (B.28)-(B.29) acquire the form

$$-\left(\frac{1}{2m_1} - \frac{1}{2m_2}\right) \lambda_3^2 M \text{const.} \tag{B.47}$$

Using Eqs. (B.35), (B.45) and (B.47) we obtain for Eq. (B.25)

$$\left[\tilde{H}_{N+1,M}(\mathbf{x}', \mathbf{y}') - \tilde{H}_{N,M}(\mathbf{x}, \mathbf{y})\right] a_k^\dagger = 2m_1 k^2 a_k^\dagger + \tag{B.48}$$
$$\text{const.} \left(\frac{\lambda_1^2}{2m_1} N(N+1) + \frac{\lambda^3}{m_2} NM + \frac{\lambda^3}{2m_1} M(M+1) - \left(\frac{1}{2m_1} - \frac{1}{2m_2}\right) \lambda_3^2 M\right) a_k^\dagger.$$

Observing that Eq. (B.32) also yields

$$\sum_{i \neq n \neq j}^{N} f(x_n - x_i) f(x_n - x_j) = \frac{1}{3} N(N-1)(N-2)\text{const.} \tag{B.49}$$

and employing (B.34) it is found that the last three lines of the auxiliary Hamiltonian $\tilde{H}_{N+1,M}(\mathbf{x}, \mathbf{y})$ in Eq. (B.24) yield the constant

$$\text{const.} \left(\frac{\lambda_1^2}{6m_1}(N+1)N(N-1) + \frac{\lambda_2^2}{6m_2} M(M-1)(M-2)\right. \tag{B.50}$$
$$\left. + \frac{\lambda_3^2}{2m_1}(N+1)M(M-1) + \frac{\lambda_3^2}{2m_2} M(N+1)N\right)$$

and similarly for $\tilde{H}_{N,M}(\mathbf{x}, \mathbf{y})$. Now Eq. (B.48) can be cast into the form

$$\left[H_{N+1,M}(\mathbf{x}', \mathbf{y}') - H_{N,M}(\mathbf{x}, \mathbf{y})\right] a_k^\dagger = 2m_1 k^2 a_k^\dagger, \tag{B.51}$$

where $H_{N,M}(\mathbf{x}, \mathbf{y})$ is defined as in (5.16) and the potential and the coupling constants are identified with

$$V(x) = f^2(x) - f'(x) + \text{const.} \quad , \tag{B.52}$$

$$g_1 = \frac{\lambda_1(\lambda_1 + \lambda^{-1})}{2m_1} \quad , \quad g_2 = \frac{\lambda_2(\lambda_2 + \lambda^{-1})}{2m_2} \quad , \quad g_3 = \frac{\lambda_3(\lambda_3 + \lambda^{-1})}{2} \left(\frac{1}{m_1} + \frac{1}{m_2}\right).$$

With $\lambda = 1$ this leads to the relations (5.42). The proof for the annihilation operator follows along the same lines. This completes the proof of part one and two of Theorem 2. In order to show part three of Theorem 2, we employ the first derivatives (B.12) and (B.13) as well as the corresponding expressions for the first derivatives with respect to the unprimed variables. Then a direct calculation leads to Eqs. (5.44) and (5.45).

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